Corrections to *On Sums Involving Fourier Coefficients of Maass Forms for* $\mathrm{SL}(3,\mathbb{Z})$

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Abstract

In our recent paper *On sums involving Fourier coefficients of Maass forms for* $\mathrm{SL}(3,\mathbb{Z})$ (Functiones et Approximatio Commentarii Mathematici, 57 (2017), 255–275), a factor $d^{2/3}$ was missing in Theorem 1. We show how to modify the proofs of the other two main results so that their conclusions are not affected by the extra factor.

Near the end of the proof of Theorem 1, when collecting main terms, a factor $d^{2/3}$ was missing. We are very grateful to Morten Risager and Anders Södergren for pointing this out. The corrected theorem statement is:

Theorem 1. *Let $x,N\in[2,\infty[$ with $N\ll x$, and let h and k be coprime integers with $1\leqslant k\leqslant x$, $k\leqslant N$ and $k\ll(Nx)^{1/3}$, the latter having a sufficiently small implicit constant depending on the underlying Maass form. Then we have*

$$\sum_{m\leqslant x}A(m,1)\,e\!\left(\frac{mh}{k}\right)$$
$$=\frac{x^{1/3}}{\pi\sqrt{3}}\sum_{d\mid k}\frac{1}{d^{1/3}}\sum_{d^2m\leqslant N_k}\frac{A(d,m)}{m^{2/3}}\,S\!\left(\overline{h},m;\frac{k}{d}\right)\cos\!\left(\frac{6\pi d^{2/3}(mx)^{1/3}}{k}\right)$$
$$+\,O(k\,x^{2/3+\vartheta+\varepsilon}\,N^{-1/3})+O(k\,x^{1/6+\varepsilon}\,N^{1/6+\vartheta}).$$

Of the other two main results of the paper, Corollary 3 is unaffected by this extra factor. Of course, on each line of the the first display of the proof, the extra factor $d^{2/3}$ must be included. Fortunately, in the end the final exponent of d is nonetheless $-1/2$ and therefore negative, and so the summation over d produces at most an extra factor $d(k)\ll k^{\varepsilon}\ll x^{\varepsilon}$.

However, in the proof of Theorem 2, and more precisely, in the proof of Lemma 11, the contribution from the off-diagonal terms suffers an extra factor $\sqrt{k}$, thereby weakening the theorem. Fortunately, with a little modification of the argument, it is possible to avoid this.

Before entering the details of the modification, let us recall that the number X_k, which comes from Lemma 10, satisfies $X_k\asymp X$, and that the Rankin–Selberg type estimate

$$\sum_{m\leqslant x}\left|A(d,m)\right|^2\ll d^2\,x,\quad\text{and its corollary}\quad\sum_{m\leqslant x}\left|A(d,m)\right|\ll d\,x,$$

which both hold uniformly for $d\in\mathbb{Z}_+$ and $x\in[1,\infty[$, imply by partial summation estimates such as

$$\sum_{m\leqslant x/d^2}\frac{\left|A(d,m)\right|^2}{m^{4/3}}\ll d^2,\qquad\sum_{m\leqslant x/d^2}\frac{\left|A(d,m)\right|^2}{m^{5/3}}\ll d^2,$$

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as well as

$$\sum_{m \leqslant x/d^2} \frac{|A(d,m)|^2}{m} \ll d^2 + d^2 \log x, \quad \text{and} \quad \sum_{m \leqslant x/d^2} \frac{|A(d,m)|}{m^{2/3}} \ll d^{1/3}\, x^{1/3}.$$

First, the treatment of the diagonal terms in the proof of Lemma 11 should be replaced by

$$\begin{aligned}
&\ll \sum_{d_1|k} \sum_{d_2|k} \sum_{d_1^2 m_1 \leqslant X_k} \sum_{\substack{d_2^2 m_2 \leqslant X_k \\ d_1^2 m_1 = d_2^2 m_2}} \frac{|A(d_1,m_1)\, A(d_2,m_2)|}{d_1^{1/3}\, d_2^{1/3}\, m_1^{2/3}\, m_2^{2/3}} \left(\frac{k^2}{d_1\, d_2}\right)^{1/2+\varepsilon} X^{5/3} \\
&\ll k^{1+\varepsilon}\, X^{5/3} \sum_{d_1|k} d_1^{-5/6} \sum_{d_2|k} d_2^{-5/6} \\
&\quad \cdot \sqrt{\sum_{d_1^2 m_1 \leqslant X_k} \sum_{\substack{d_2^2 m_2 \leqslant X_k \\ d_1^2 m_1 = d_2^2 m_2}} \frac{|A(d_1,m_1)|^2}{m_1^{4/3}}} \sqrt{\sum_{d_1^2 m_1 \leqslant X_k} \sum_{\substack{d_2^2 m_2 \leqslant X_k \\ d_1^2 m_1 = d_2^2 m_2}} \frac{|A(d_2,m_2)|^2}{m_2^{4/3}}} \\
&\ll k^{1+\varepsilon}\, X^{5/3} \sum_{d_1|k} d_1^{-5/6} \sum_{d_2|k} d_2^{-5/6} \sqrt{d_1^2\, d_2^2} \\
&\ll k^{1+\varepsilon}\, X^{5/3} \sum_{d_1|k} d_1^{1/6} \sum_{d_2|k} d_2^{1/6} \ll k^{4/3+\varepsilon}\, X^{5/3},
\end{aligned}$$

where it is useful to observe that the sum under the first square root sign is empty unless $d_1 \leqslant \sqrt{X_k}$, and similarly for the second sum.

In the treatment of the off-diagonal terms, we split the sum over m_2 into two parts by

$$\sum_{d_1^2 m_1 < d_2^2 m_2 \leqslant X_k} = \sum_{d_1^2 m_1 < d_2^2 m_2 < \min\left\{X_k + 1, d_1^2 m_1 + d_2^2/2\right\}} + \sum_{d_1^2 m_1 + d_2^2/2 \leqslant d_2^2 m_2 \leqslant X_k}.$$

The second set of off-diagonal terms can be estimated basically in the same way as in the paper by first estimating the integral with the first derivative test, leading to the contribution

$$\begin{aligned}
&\ll \sum_{d_1|k} \sum_{d_2|k} \sum_{d_1^2 m_1 \leqslant X_k} \sum_{d_1^2 m_1 + d_2^2/2 \leqslant d_2^2 m_2 \leqslant X_k} \frac{|A(d_1,m_1)\, A(d_2,m_2)|}{d_1^{1/3}\, d_2^{1/3}\, m_1^{2/3}\, m_2^{2/3}} \\
&\quad \cdot \left(\frac{k^2}{d_1 d_2}\right)^{1/2+\varepsilon} \frac{k\, X^{4/3}\, (d_2^{2/3}\, m_2^{1/3})^2}{|d_1^2\, m_1 - d_2^2\, m_2|} \\
&\ll k^2\, X^{4/3+\varepsilon} \sum_{d_1|k} d_1^{-5/6} \sum_{d_2|k} d_2^{1/2} \\
&\quad \cdot \sum_{d_1^2 m_1 \leqslant X_k} \frac{|A(d_1,m_1)|}{m_1^{2/3}} \sum_{d_1^2 m_1 + d_2^2/2 \leqslant d_2^2 m_2 \leqslant X_k} \frac{|A(d_2,m_2)|}{d_2^2 m_2 - d_1^2 m_1} \\
&\ll k^2\, X^{4/3+\varepsilon} \sum_{d_1|k} d_1^{-5/6} \sum_{d_2|k} d_2^{1/2} \sum_{d_1^2 m_1 \leqslant X_k} \frac{|A(d_1,m_1)|}{m_1^{2/3}}\, d_2^{-2}\, d_2^{\vartheta+\varepsilon} \left(\frac{X}{d_2^2}\right)^{\vartheta+\varepsilon} \\
&\ll k^2\, X^{4/3+\varepsilon} \sum_{d_1|k} d_1^{-5/6} \sum_{d_2|k} d_2^{1/2} d_1^{1/3}\, X^{1/3}\, d_2^{-2-\vartheta}\, X^{\vartheta+\varepsilon} \\
&\ll k^2\, X^{5/3+\vartheta+\varepsilon} \sum_{d_1|k} d_1^{-1/2} \sum_{d_2|k} d_2^{-3/2-\vartheta} \ll k^2\, X^{5/3+\vartheta+\varepsilon},
\end{aligned}$$

where, in particular, we estimate

$$\sum_{d_1^2 m_1 + d_2^2/2 \leqslant d_2^2 m_2 \leqslant X_k} \frac{1}{d_2^2 m_2 - d_1^2 m_1} \ll \sum_{1 \leqslant \nu \leqslant X_k/d_2^2} \frac{1}{d_2^2 \nu} \ll d_2^{-2} \log X.$$

In the first set of off-diagonal terms, we do not apply the first derivative test, but instead estimate the integral by absolute values as in the diagonal terms, leading to

$$\begin{aligned}
&\ll \sum_{d_1 | k} \sum_{d_2 | k} \sum_{d_1^2 m_1 \leqslant X_k} \sum_{\substack{d_2^2 m_2 \leqslant X_k, \\ d_1^2 m_1 < d_2^2 m_2 < d_1^2 m_1 + d_2^2/2}} \frac{|A(d_1, m_1)\, A(d_2, m_2)|}{d_1^{1/3}\, d_2^{1/3}\, m_1^{2/3}\, m_2^{2/3}} \\
&\qquad \cdot \left(\frac{k^2}{d_1\, d_2} \right)^{1/2+\varepsilon} X^{5/3} \\
&\ll k^{1+\varepsilon}\, X^{5/3} \sum_{d_1 | k} d_1^{-5/6} \sum_{d_2 | k} d_2^{-5/6} \\
&\qquad \cdot \sqrt{\sum_{d_1^2 m_1 \leqslant X_k} \sum_{\substack{d_2^2 m_2 \leqslant X_k, \\ d_1^2 m_1 < d_2^2 m_2 < d_1^2 m_1 + d_2^2/2}} \frac{|A(d_1, m_1)|^2}{m_1}} \\
&\qquad \cdot \sqrt{\sum_{d_1^2 m_1 \leqslant X_k} \sum_{\substack{d_2^2 m_2 \leqslant X_k, \\ d_1^2 m_1 < d_2^2 m_2 < d_1^2 m_1 + d_2^2/2}} \frac{|A(d_2, m_2)|^2}{m_1^{1/3}\, m_2^{4/3}}} \\
&\ll k^{1+\varepsilon}\, X^{5/3} \sum_{d_1 | k} d_1^{-5/6} \sum_{d_2 | k} d_2^{-5/6} \sqrt{d_1^2 \log X}\, \sqrt{d_1^{2/3}\, d_2^{4/3}\, \max\left\{ d_2^2\, d_1^{-2}, 1 \right\}} \\
&\ll k^{1+\varepsilon}\, X^{5/3+\varepsilon} \left(\sum_{d_1 | k} d_1^{1/2} \sum_{d_2 | k} d_2^{-1/6} + \sum_{d_1 | k} d_1^{-1/2} \sum_{d_2 | k} d_2^{5/6} \right) \\
&\ll k^{11/6}\, X^{5/3+\varepsilon}.
\end{aligned}$$

Here the sum involving $|A(d_2, m_2)|^2$ is estimated by first observing that $d_1^2 m_1 \asymp d_2^2 m_2$, so that

$$\begin{aligned}
&\sum_{d_1^2 m_1 \leqslant X_k} \sum_{\substack{d_2^2 m_2 \leqslant X_k, \\ d_1^2 m_1 < d_2^2 m_2 < d_1^2 m_1 + d_2^2/2}} \frac{|A(d_2, m_2)|^2}{m_1^{1/3}\, m_2^{4/3}} \\
&\qquad\qquad \ll d_1^{2/3}\, d_2^{-2/3} \sum_{d_1^2 m_1 \leqslant X_k} \sum_{\substack{d_2^2 m_2 \leqslant X_k, \\ d_1^2 m_1 < d_2^2 m_2 < d_1^2 m_1 + d_2^2/2}} \frac{|A(d_2, m_2)|^2}{m_2^{5/3}},
\end{aligned}$$

and when estimating the double sum over m_1 and m_2 we observe that for each value of m_1 there is at most one value of m_2 with $d_2^2 m_2 \leqslant X_k$ satisfying the inequalities $d_1^2 m_1 < d_2^2 m_2 < d_1^2 m_1 + d_2^2/2$. Furthermore, one such specific value of m_2 can occur for at most $\ll \max\left\{ d_2^2/d_1^2, 1 \right\}$ different values of m_1 and so the estimations may be continued by

$$\begin{aligned}
&\ll d_1^{2/3}\, d_2^{-2/3} \max\left\{ d_2^2\, d_1^{-2}, 1 \right\} \sum_{d_2^2 m_2 \leqslant X_k} \frac{|A(d_2, m_2)|^2}{m_2^{5/3}} \\
&\ll d_1^{2/3}\, d_2^{-2/3} \max\left\{ d_2^2\, d_1^{-2}, 1 \right\} d_2^2 = d_1^{2/3}\, d_2^{4/3} \max\left\{ d_2^2\, d_1^{-2}, 1 \right\}.
\end{aligned}$$