Eigenvalues of Periodic Magnetic Schrödinger Operators

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Abstract

The spectral theory of Schrödinger operators with periodic potentials arose from the nearly-free electron approximation of solid state physics. One of the central questions of the topic concerns the nature of the spectra of such operators. In the self-adjoint case, one expects them to be absolutely continuous. This turns out to be equivalent to absence of eigenvalues, which is also expected to hold in the non-self-adjoint case.

After some introductory words, we give an exposition of L. E. Thomas' approach to proving absence of eigenvalues, which basically uses a Floquet decomposition to reduce the task of establishing absence of eigenvalues for $(-i\nabla - A)^2 + V$ with periodic potentials to obtaining a value of $k \in \mathbb{C}^n$ for which the operator

$$(-i\nabla - A + 2\pi k)^2 + V\colon \mathscr{H}^2(\mathbb{T}^n) \longrightarrow \mathscr{L}^2(\mathbb{T}^n) \qquad (*)$$

is injective. It turns out that this can be easily proved when A is sufficiently small.

The remainder is dedicated to an exposition of A. V. Sobolev's theorem which first established absolute continuity of the magnetic Schrödinger operator $(-i\nabla - A)^2 + V$ given that A is sufficiently smooth. Among other things, we shall describe a proof for the following special case:

Theorem. *Let $n \in \mathbb{Z}_+$, let $V \in \mathscr{L}^\infty(\mathbb{R}^n)$ be periodic, let $A \in \mathscr{C}^\infty(\mathbb{R}^n; \mathbb{C}^n)$ have trigonometric polynomials as its components, and assume that $\widehat{A}(0) = 0$. Then the operator*

$$(-i\nabla - A)^2 + V\colon \mathscr{H}^2(\mathbb{R}^n) \longrightarrow \mathscr{L}^2(\mathbb{R}^n)$$

has no eigenvalues.

The main idea in Sobolev's proof is the construction of toroidal pseudodifferential operators such that the first-order terms can be made arbitrarily small by conjugating the operator $(*)$ with them.

Naturally, we also give a minimal sufficient background of the required supporting tools; the modified Floquet transform, analytic families of operators of type $\mathscr{A}$, the analytic Fredholm theorem, and the parameter-dependent toroidal pseudodifferential operators.

Introduction

We begin with a nontechnical description of some classic problems of solid state physics and the nearly-free electron approximation, which first gave rise to periodic Schrödinger operators. We then move on to a very brief general discussion of spectral theory of periodic elliptic operators. Due to the wealth of research conducted on them, a comprehensive treatment is hopelessly beyond our possibilities here, and we shall not attempt such a feat. Finally, we finish with some general remarks concerning the rest of the text.

The primary source for the discussion on solids was the textbook [Tan] by B. K. Tanner. The textbooks of Arfken & Mermin [A&M], Hook & Hall [Ho&Ha], and Kittel [Ki] are standard references, and were also quite useful.

The discussion of the spectral theory of periodic operators is primarily based on the survey article [Bi&S3] of M. Sh. Birman and T. A. Suslina. Other useful sources for general information and references are the articles by P. Kuchment [Ku2], Kuchment and S. Levendorskiĭ [Ku&L2], A. V. Sobolev [So2], and Suslina [Su].

Background: the physics of crystalline solids

It is a fact of nature that in most solid materials, the atoms are locally arranged in highly regular arrays. This had been already suspected on the basis of the macroscopic symmetry of crystals, but was demonstrated definitively with the first X-ray diffraction experiments in 1912. For example, engineering steels are composed of roughly micrometer size crystals, in each of which the atoms are arranged in a lattice, modulo impurities. The regular patterns involved may be highly complex, however. In some organic solids, the number of atoms in a period cell may even reach many thousands of atoms. Of course, some materials, like glasses, do not exhibit such long-range order, but much of solid state physics has always been concerned with crystalline solids.

Solid materials exhibit many intriguing phenomena which one would like to be able to explain. For instance, classical physics arrived at the conclusion that electrical current is caused by nearly-free movement of charged particles, typically electrons, in the material. This notion was further amplified by the early successes in explaining some properties of metals using the free-electron models of solids, namely the Drude theory and the Sommerfeld theory, which modeled electrons in a solid as classical gas and as a quantum gas, respectively. However, X-ray diffraction experiments taught that in solid materials the atoms, the size of which are roughly of the order of 1 Å, are spaced some, also roughly, few Ångströms apart. This raises the question: just how do the charged particles move in such a cluttered environment.

There are other obvious questions one might ask, too: Why are there electrical conductors and insulators? What is the essential difference between the two? The free-electron models can not possibly account for the fact that insulators exist at all. Why are there semiconducting materials and what are they? The existence of electrical conductors and insulators is made very striking by the fact that the differences in conductivities of materials are larger than in any other properties of materials; the orders of magnitudes of best conductors and best insulators differ roughly by 10^{30}.

The nearly-free electron approximation

Since it is not possible, for obvious reasons, to use Schrödinger equation to model each individual electron in an object, some drastic simplifications are needed. The nearly-free electron approximation makes, among others, the following:

1. The atoms of the material form a perfect infinite lattice, and are perfectly stationary.
2. An individual electron "feels" the other electrons as a periodic perturbation of the electric potential.

In other words, the nearly-free electron approach models an electron in a solid using a Schrödinger operator $-\Delta + V$ with a periodic electric potential V. In a sense, the nearly-free electron approximation represents the next level of complexity after Sommerfeld's model.

Even though the assumptions made are not quite realistic, the resulting model is quite powerful. As an example, we will very briefly describe how this model qualitatively accounts for the differences of conductors, insulators and semiconductors.

The possible energies of a free electron, corresponding to the free Schrödinger operator $-\Delta$, form a straightforward continuum $[0, \infty[$. The introduction of the periodic perturbation V "opens up" gaps in this spectrum. This is called the band structure. The intervals arise naturally from the operator $-\Delta + V$ itself. In fact the operator $-\Delta + V$ can be broken into a family of operators H_k on the torus for $k \in [0, 1[^n$, each H_k having discrete spectrum. More precisely, the spectrum of H_k consists of an increasing sequence of eigenvalues

$$\lambda_1(k) \leqslant \lambda_2(k) \leqslant \lambda_3(k) \leqslant \dots$$

tending to infinity.

The band structure follows from the fact that each eigenvalue $\lambda_\ell(k)$, where $\ell \in \mathbb{Z}_+$, depends continuously on k. In particular, the ℓ^{th} spectral band will be the closure of the image of ℓ^{th} eigenvalues $\lambda_\ell(\cdot)$. The manner in which the operators H_k are obtained is not quite unique. Our mathematical discussion will begin by the introduction of the modified Floquet transform, which will achieve a decomposition suitable for us.

Given the band structure the difference between conductors and insulators arises in the following way. Let us consider an even more idealized absolute zero temperature situation, in which the electrons in a crystalline solid will occupy the energy bands according to the principle of Pauli up to a certain energy level called the Fermi energy.

Now, if the Fermi energy is located at the bottom of a wide spectral gap, then in order to get conduction, one has to give each conduction electron enough energy to get them past the gap, and naturally this would be difficult. That is, in this situation the solid material is an insulator.

On the other hand, if the Fermi energy is located in the middle of a spectral band, then an infinitesimal push is enough to get conduction electrons. Also, if the Fermi energy is at the bottom of a gap small enough that a relatively significant amount of electrons can be thermally excited to the conduction band at room temperature, the material is a semiconductor. Silicon would be a typical example.

Even though the nearly-free electron models are rather impressive, they too are still crude approximations. For instance, nearly-free electron approximation ignores the fact that the atoms of the lattice vibrate, and these lattice vibrations need to be invoked in order to explain heat conduction in insulators. Other examples of phenomena with which the nearly-free electron approximation breaks down include magnetization and superconductivity.

Eigenvalues of periodic Schrödinger operators

Our topic really begins with a certain paper [Th] of L. E. Thomas from 1973. For preceding work, we refer to the monograph [Ea] of M. S. P. Eastham and references therein. In fact, in [Ea], the contemporary spectral theory of the periodic Schrödinger operator $-\Delta + V$ is described as "somewhat sparse".

The issue of from what classes one should take the potentials is somewhat messy, various $\mathscr{L}^p$-spaces, Lorentz spaces, Kato classes and other possibilities making appearance, and there is a non-trivial dependence on the dimension. We do not want to get bogged down by such details. Instead we mention the discussion of more precise conditions in [Bi&S3], and for further details the original papers may also be consulted.

Thomas' 1973 paper [Th]

In his paper Thomas proved that for a real-valued locally $\mathscr{L}^2$ periodic electric potentials V, the three-dimensional Schrödinger operator $-\Delta + V$ has absolutely continuous spectrum. The motivation behind this was to establish a sound basis for studying the scattering from local impurities in a crystal, more precisely establishing the meaningfulness of the relevant wave operators. The importance of this paper stems largely from the fact that Thomas devised a general approach to proving absolute continuity for periodic operators, an approach which in fact has been employed in essentially all of the subsequent periodic absolute continuity proofs.

We pause to remark that in the self-adjoint case, the absolute continuity of the spectrum of a periodic operator is essentially equivalent to the absence of eigenvalues, as the emptiness of the singular continuous spectrum is not hard to establish. (See e.g. [R&S2, sect. XIII.16].) Since we have no reason to restrict ourselves to the self-adjoint case, and since the methods with which we shall be concerned do not depend on self-adjointness, we prefer to think in terms of eigenvalues instead of the absolutely continuous spectrum.

Of course, Thomas' approach was generalized to arbitrary dimensions. The first printed account appeared in section XIII.16 of M. Reed and B. Simon's volume [R&S2]. The conditions on V were not optimal, and have improved substantially in subsequent expositions and papers of which we mention those of Birman and Suslina [Bi&S3] and Zh. Shen [She1, She2, She3].

More general Schrödinger operators

The next chapter in the story of eigenvalues of periodic operators begins with the observation of R. Hempel and I. Herbst in [He&He1, He&He2] that the periodic magnetic Schrödinger operator $(-i\nabla - A)^2 + V$, where A is a periodic vector-valued potential, has no eigenvalues when the components of A are small in the $\mathscr{L}^\infty$-norm.

M. Sh. Birman and T. A. Suslina established the absolute continuity of the magnetic Schrödinger operator in the two-dimensional case in [Bi&S1, Bi&S2]. A. Morame [Mo] established the absence of eigenvalues for general two-dimensional periodic elliptic second-order partial differential operators, assuming the coefficients are sufficiently smooth.

The two-dimensional result in [Bi&S1] motivated A. V. Sobolev to study the problem for magnetic Schrödinger operators in higher dimensions. He succeeded in proving the higher-dimensional absence of eigenvalues in [So1] by constructing invertible zeroeth-order pseudodifferential operators and conjugating the original Schrödinger operator with them in order to transform the first-order terms to be so small that the ideas of Thomas and Hempel–Herbst apply.

Soon after Sobolev's work, in [Ku&L1, Ku&L2] P. Kuchment and S. Levendorskiĭ brought the proof of Sobolev to handle more or less all classes of periodic operators, self-adjoint and non-self-adjoint, scalar-valued or vector-valued, for which absence of eigenvalues had been proved, given sufficiently smooth coefficients.

We should remark that similar pseudodifferential conjugation has been used to remove large first-order terms in a variety of topics, such as inverse problems for partial differential equations [N&U1, N&U2, Sa1], inverse scattering theory [Es&R, I,N&U], as well as in nonlinear Schrödinger equations [Tak, Ke,P&V].

The general conjecture

The results mentioned above lead to the conjecture that any periodic elliptic second-order partial differential operator should have no eigenvalues. This is now known in one and two dimensions, and in higher dimensions for magnetic Schrödinger operators. For consideration of the smoothness of coefficients, we refer to Birman and Suslina's survey [Bi&S3].

A breakthrough towards the proof of this conjecture in higher dimensions appeared in L. Friedlander's paper [Fr] where he proved that if the coefficients of a self-adjoint second-order elliptic periodic partial differential operator are smooth and even in one variable, say x_1, then the operator is absolutely continuous. The reason this was a breakthrough is that before Friedlander's result all absence of eigenvalue results that appeared to handle more general metrics essentially reduced those cases to the case of constant metric (see e.g. the discussion in [Ku&L2]). That is, Friedlander was the first to handle truly general metrics and to prove the conjecture under the evenness assumption.

Higher-order operators

Of course, there is no clear mathematical a priori reason to focus merely on second-order periodic operators, and it is natural to ask whether one might reasonably hope for a more general absence of eigenvalues result. However, the answer seems to be negative. The main reason why this should be so is the failure of unique continuation for higher-order operators, allowing some linear elliptic operators to have compactly supported non-zero solutions.

In particular, in the paper [P1], A. Pliś constructed a three-dimensional fourth-order elliptic equation with smooth coefficients of the form

$$(1-\chi)\,\Delta^2 u + \chi E u = 0,$$

where $\chi \in \mathscr{C}_c^\infty\left(\mathbb{R}^3\right)$ and E is some linear elliptic fourth-order partial differential operator with smooth coefficients, with a non-zero solution $u \in \mathscr{C}_c^\infty\left(\mathbb{R}^3\right)$. It is obvious how this gives rise to a periodic operator with an eigenvalue.

One might also hope for absence of eigenvalues result for periodic operators of more restrictive form, such as $(-\Delta)^k + L$, where L would be a periodic partial differential operator of order smaller than $2k$. However, in his treatise [C], P. Cohen constructed an equation of the form $(-\Delta)^3 u + Lu = 0$, where L is a fifth-order linear differential operator, with a non-zero solution $u \in \mathscr{C}_c^\infty(\mathbb{R}^n)$. The coefficients of Cohen's counter-example were not infinitely smooth but continuous. On the other hand, S. Mizohata [Mi] has proven a unique continuation result for fourth-order operators of the form $L_1L_2 + L_3$, where L_1 and L_2 are second-order elliptic operators with smooth real coefficients, and L_3 is a third-order operator with smooth coefficients.

Finally, we mention that unique continuation fails even for second-order elliptic equations having only Hölder continuous coefficients, as was first proven by Pliś in [P2]. Recently, in [Fi] N. Filonov constructed a Hölder continuous positive-definite real-valued metric γ, such that the equation $-\nabla \cdot \gamma \nabla u = \lambda u$ has a compactly supported non-zero solution for some $\lambda \in \mathbb{R}$.

Other aspects of spectral theory of periodic operators

The spectral theory of periodic operators is a vast subject and we have pretty much neglected most of it. Among other things, there has been quite a lot work on the absolute continuity problem of various vector-valued operators, such as Dirac operators, Pauli operators and Maxwell operators (see e.g. [Bi&S4, Ku&L2] and references in them). The periodic Maxwell operators in particular are highly relevant in the recent field of photonic crystals. For the physics we refer to the textbook of J. D. Joannopoulos et al. [J&al.]. The mathematics is discussed in Kuchment's survey [Ku2].

In the scalar self-adjoint case there are many other deep topics, of which we restrict ourselves to mention only one. The Bethe–Sommerfeld conjecture basically says that in the multidimensional case, the spectrum of a periodic self-adjoint operator, which is just a union of countably many closed intervals, should contain a half-line. In other words, in the multidimensional case, the spectrum of a periodic operator should have only finitely many gaps. For a survey on this topic, we refer to Sobolev's [So2]. We would like to mention the following result, though: It was proved in [P&S] by L. Parnovski and Sobolev that given $m \in \mathbb{R}_+$ and a periodic pseudodifferential operator B of order smaller than $2m$, the periodic operator $(-\Delta)^m + B$ satisfies the conclusion of the Bethe–Sommerfeld conjecture.

On this work

The structure of this text is most easily seen simply by taking a quick look at the table of contents. However, the following remarks might clarify some issues.

The next chapter contains a brief discussion of the notation used, and gives also an impression of the prerequisites. For sake of completeness, we assume that the reader is familiar with basic Fourier analysis, including both Fourier series and Fourier transforms, basic function theory, Sobolev spaces $\mathscr{H}^s$ ($s \in \mathbb{R}_+$), and the basic functional analysis, including Hilbert spaces and their bounded and unbounded linear operators, and the very basics of corresponding spectral theory. No deep results, such as the spectral theorem, will be used.

The chapters on perturbation theory and toroidal pseudodifferential operators are self-contained and independent of the other chapters. Only some basic concepts and results need to be known from them in order to read the rest of the text.

The main thread consists of the three chapters on Floquet decompositions, Thomas' argument and Sobolev's argument, which are recommended to be read in this order. The order of chapters reflects the logical ordering.

In general, the relevant references and sources are given at the beginning of each chapter, though in line references may sometimes appear.

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Notation

The symbol $\mathbb{Z}_+$ stands for the set $\{1, 2, 3, \ldots\}$. The letters $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ have their usual meaning.

The letter n always denotes a fixed positive integer. Everything may be taken to depend on n. The n-dimensional euclidean space is denoted by $\mathbb{R}^n$ and the set of all n-tuples of integers by $\mathbb{Z}^n$. The symbol $\mathbb{T}^n$ denotes the n-torus, which may be viewed as the unit cube $[0,1]^n$ with opposite faces glued together in the obvious manner, or rather as the quotient space $\mathbb{R}^n/\mathbb{Z}^n$. The letter Q shall denote the cube $[0,1[^n$.

We reiterate that a 1-periodic function means for us a function f from $\mathbb{R}^n$ to some non-empty set, satisfying

$$f(x+\xi) = f(x)$$

for all $x \in \mathbb{R}^n$ and every $\xi \in \mathbb{Z}^n$. If the lattice $\mathbb{Z}^n$ was to be replaced by some other lattice Γ, the corresponding concept would be that of Γ-periodicity. Since no other lattices ever make appearance, we always say periodic instead of 1-periodic.

The problem of defining $\mathbb{T}^n$ may be avoided quite easily, as we will always be interested in functions defined on $\mathbb{T}^n$ and never in the geometry or topology of $\mathbb{T}^n$. Therefore, all functions defined on $\mathbb{T}^n$ may simply be construed to be periodic functions on $\mathbb{R}^n$.

We use the useful shorthand notation $e(\cdot)$ for $e^{2\pi i \cdot}$. The angle brackets $\langle\cdot\rangle$ denote $\sqrt{1+|\cdot|^2}$, as usual.

For a vector $x \in \mathbb{R}^n$, we write x^2 for $|x|^2 = x \cdot x$. For example, we might say $\langle x\rangle = \sqrt{1+x^2}$. For complex vectors $z = \langle z_1, z_2, \ldots, z_n\rangle \in \mathbb{C}^n$, we write

$$z^2 = z_1^2 + z_2^2 + \ldots + z_n^2.$$

This is usually not equal to $|z|^2$, and certainly not necessarily real-valued.

Given a point $z_0 \in \mathbb{C}$ and a radius $\varepsilon \in \mathbb{R}_+$, we write

$$\begin{cases} B(z_0,\varepsilon) = \left\{ z \in \mathbb{C} \big| \, |z - z_0| < \varepsilon \right\}, \\ \overline{B}(z_0,\varepsilon) = \left\{ z \in \mathbb{C} \big| \, |z - z_0| \leqslant \varepsilon \right\}, \quad \text{and} \\ \partial B(z_0,\varepsilon) = \left\{ z \in \mathbb{C} \big| \, |z - z_0| = \varepsilon \right\}. \end{cases}$$

The boundary $\partial B(z_0, \varepsilon)$ will occasionally we interpreted as a curve oriented to the usual direction.

The $\mathscr{L}^p$-spaces are denoted, as usual, by $\mathscr{L}^p(\Omega)$ for $p \in [1,\infty]$, where Ω will always be either $\mathbb{R}^n$ or $\mathbb{T}^n$, and p will always be either 2 or ∞. We denote the

$\mathscr{L}^p$-space of $\mathbb{C}^n$-valued functions by $\mathscr{L}^p(\Omega;\mathbb{C}^n)$. We employ similar notation for Sobolev spaces (see below). Given a Hilbert space $\mathscr{H}$, we denote the $\mathscr{L}^2$-space of $\mathscr{H}$-valued functions by $\mathscr{L}^2(\Omega;\mathscr{H})$. The precise nature of these spaces is the first topic of the next chapter.

Multi-index notation

Given the dimension $n \in \mathbb{Z}_+$, an n-dimensional multi-index α is an n-tuple $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$ of non-negative integers. We shall write

$$|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n, \quad \text{and} \quad \alpha! = \alpha_1! \cdot \alpha_2! \cdot \ldots \cdot \alpha_n!.$$

This use of $|\cdot|$ does conflict with our notation for the Pythagorean length of a vector, but in practice there is never a danger of confusion.

Given another multi-index β, we also write $\beta \leqslant \alpha$, if $\beta_\ell \leqslant \alpha_\ell$ for each $\ell \in \{1, 2, \ldots, n\}$. The relation $\beta < \alpha$ is defined similarly. When $\beta \leqslant \alpha$ we also write

$$\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2}\cdots\binom{\alpha_n}{\beta_n}.$$

The symbols

$$\sum_{|\alpha|=N}, \quad \sum_{|\alpha|<N}, \quad \sum_{\beta\leqslant\alpha}, \quad \sum_{\beta<\alpha}, \quad \text{and} \quad \sum_{|\beta|\leqslant|\alpha|},$$

where $N \in \mathbb{Z}_+$, will have their obvious meanings. Finally, we employ the following space saving notation for derivatives:

$$\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}.$$

Fourier analysis and Sobolev spaces

In our text the Fourier transform is defined by the formula

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e(-x\cdot\xi)\,\mathrm{d}\xi$$

for all $\xi \in \mathbb{R}^n$ and every Schwartz function $f \in \mathscr{S}(\mathbb{R}^n)$. We recall that this may be extended to a Hilbert space isomorphism from $\mathscr{L}^2(\mathbb{R}^n)$ to itself. Given $s \in [0,\infty[$, the Sobolev space $\mathscr{H}^s(\mathbb{R}^n)$ is defined as follows:

$$\mathscr{H}^s(\mathbb{R}^n) = \left\{ f \in \mathscr{L}^2(\mathbb{R}^n) \,\middle|\, |\cdot|^s\,\widehat{f} \in \mathscr{L}^2(\mathbb{R}^n) \right\}.$$

For a periodic function $f\colon \mathbb{T}^n \longrightarrow \mathbb{C}$, we define the Fourier coefficients via the formula

$$\widehat{f}(\xi) = \int_{\mathbb{T}^n} f(x)\, e(-x\cdot\xi)\,\mathrm{d}x$$

for each $\xi \in \mathbb{Z}^n$. Of course, this gives rise to a Hilbert space isomorphism from $\mathscr{L}^2(\mathbb{T}^n)$ to $\ell^2(\mathbb{Z}^n)$. Given $s \in [0,\infty[$, the Sobolev space $\mathscr{H}^s(\mathbb{T}^n)$ is the space of functions $f \in \mathscr{L}^2(\mathbb{T}^n)$ for which

$$\sum_{\xi\in\mathbb{Z}^n} |\xi|^{2s} \big|\widehat{f}(\xi)\big|^2 < \infty.$$

Hilbert and Banach spaces

When discussing general Banach spaces, the norm is always denoted by $\|\cdot\|$. Any Hilbert space considered will be infinite-dimensional and separable, and the corresponding inner product is denoted by $\langle\cdot|\cdot\rangle$, which is always to be taken to be linear in the second variable, and conjugate linear in the first.

The Banach space of bounded linear operators of a Banach space $\mathscr{X}$ to itself is denoted by $\mathscr{B}(\mathscr{X})$. The operator norm is denoted by $\|\cdot\|$, with some refining subindex, if necessary.

The class of closed operators of a Hilbert space $\mathscr{H}$ is denoted by $\mathscr{C}(\mathscr{H})$. The domain of a densely defined linear operator A of $\mathscr{H}$ is denoted by $\operatorname{Dom} A$. The resolvent set and the spectrum of a closed linear operator A are denoted by $\rho(A)$ and $\sigma(A)$, respectively.

Asymptotic notation

For asymptotic relations we use the standard symbols O, o, Ω, $\ll$, $\gg$, Θ, etc. Let f and g be functions with equal domains of definition. Then $O(g)$ and $\Omega(g)$ denote any complex valued quantities that are bounded in modulus from above and, respectively, below by the expression $C|g|$ in the domain of definition of g for some implicit constant $C \in \mathbb{R}_+$. Such a quantity is $\Theta(g)$ if it is both $O(g)$ and $\Omega(g)$.

Similarly, we write $f \ll g$ to signify that f is $O(g)$, $f \gg g$ to signify that f is $\Omega(g)$, and $f \asymp g$ to signify that f is $\Theta(g)$. If the inequalities related to O, Θ, Ω, $\ll$, $\gg$ or $\asymp$ are to hold only for sufficiently large values of some parameter τ, we append the expression $(\tau \longrightarrow \infty)$ to the formulas.

All the implicit constants will depend on the dimensions $n \in \mathbb{Z}_+$, and the potentials W and V. When the constants are allowed to depend on some other parameters, those parameters are given in subindices to O, Θ and Ω. As an exception, when discussing the pseudodifferential operators, the implicit constants will typically depend also on the symbols considered.

Floquet Decompositions

In this chapter we describe a certain Hilbert space isomorphism

$$\mathscr{F}\!\ell\colon \mathscr{L}^2(\mathbb{R}^n) \cong \mathscr{L}^2\big(Q; \mathscr{L}^2(Q)\big)\,,$$

called the (modified) **Floquet transform**. Here Q denotes the unit cube $[0,1[^n$. The reasons for introducing this transform are the following:

- for any $\varphi \in \mathscr{H}^2(Q)$, we have

  $$\big(\mathscr{F}\!\ell\,\varphi\big)(k) \in \mathscr{H}^2(\mathbb{T}^n)$$

  for almost every $k \in Q$; and
- furthermore, for almost every $k \in Q$,

  $$\big(\mathscr{F}\!\ell\,(-\Delta - iW\cdot\nabla + V)\,\varphi\big)(k) = (\Delta_k + W\cdot\nabla_k + V)\big(\mathscr{F}\!\ell\,\varphi\big)(k)\,,$$

  where ∇_k is the vector-valued differential operator $-i\nabla + 2\pi k$, Δ_k denotes the scalar operator $\nabla_k^2 = (-i\nabla + 2\pi k)^2$, and W and V are periodic functions, say from the classes $\mathscr{L}^\infty(\mathbb{R}^n;\mathbb{C}^n)$ and $\mathscr{L}^\infty(\mathbb{R}^n)$, respectively.

That is, questions concerning the spectrum of the periodic magnetic Schrödinger operator $-\Delta - iW\cdot\nabla + V$ are turned to questions concerning the spectra of the operators $\Delta_k + W\cdot\nabla_k + V$, each having $\mathscr{H}^2(\mathbb{T}^n)$ as its domain. We will see later that these operators give rise to analytic families of type $\mathscr{A}$ with compact resolvents, and such families of operators have many good properties.

The basics of Floquet decompositions, using the language of direct integrals, can be found in the section XIII.16 of Reed and Simon's [R&S2]. Another standard reference, a much more general one, is Kuchment's monograph [Ku1].

What is $\mathscr{L}^2\big(Q; \mathscr{L}^2(Q)\big)$?

Before we consider the Floquet transform, we have to define certain spaces, namely, $\ell^2\big(\mathbb{Z}^n; \mathscr{L}^2(Q)\big)$, $\mathscr{L}^2\big(Q; \ell^2(\mathbb{Z}^n)\big)$, and $\mathscr{L}^2\big(Q; \mathscr{L}^2(Q)\big)$. The first of these is unproblematic as it is obvious how to define it and why it must be a Hilbert space. The second space is also easy to define adequately: we just choose the elements of $\mathscr{L}^2\big(Q; \ell^2(\mathbb{Z}^n)\big)$ to be (equivalence classes of) functions from Q to $\ell^2(\mathbb{Z}^n)$ such that the canonical coordinate functions are measurable and that the obvious $\mathscr{L}^2\big(Q; \ell^2(\mathbb{Z}^n)\big)$-norm is finite.

In the case of the third space, the norm and inner product are again straightforward to define. However, we now face the problem of what it means for a function f from Q to $\mathscr{L}^2(Q)$ to be measurable? On one hand, we want to say that f is measurable precisely when the inverse images $f^{-1}[O]$ of open sets O in $\mathscr{L}^2(Q)$ are measurable in Q. On the other hand, we also would like f to be measurable exactly when its Fourier coefficients are measurable, or more generally, exactly when the function $\langle \varphi | f(\cdot) \rangle$ is measurable on Q for any fixed vector $\varphi \in \mathscr{L}^2(Q)$. Fortunately, it turns out that for functions taking values in a Hilbert space, the three natural concepts of measurability turn out to be equivalent. That is, we have the following result which we present following the appendix to the section IV.5 in [R&S1].

Proposition. *For a function $f\colon Q \longrightarrow \mathscr{H}$, where $\mathscr{H}$ is just some arbitrary separable infinite-dimensional Hilbert space, the following three conditions are equivalent:*

a) the function f is almost everywhere the pointwise limit of a sequence of simple functions from Q to $\mathscr{H}$;

b) for any open set U of $\mathscr{H}$, the inverse image $f^{-1}[U]$ is measurable;

c) for any $\varphi \in \mathscr{H}$, the function

$$x \longmapsto \langle \varphi | f(x) \rangle : Q \longrightarrow \mathbb{C}$$

is measurable.

Some remarks are in order. First, we recall that a simple function means a function taking only finite many values, and taking each of them only in a measurable set. Second, the condition *c*) is clearly equivalent with the condition that the coordinate functions of f with respect to some fixed orthonormal basis of $\mathscr{H}$ are measurable. Third, in [R&S1, p. 116] the notions of measurability defined by the conditions a) and c) are called **strong** measurability and **weak** measurability, respectively.

Proof of the proposition. a)$\Longrightarrow$b). Let $\langle f_k \rangle_{k=1}^{\infty}$ be a sequence of functions from Q to $\mathscr{H}$ such that for each $\ell \in \mathbb{Z}_+$, f_ℓ takes only finitely many values, and each of these values is taken precisely on a measurable subset of Q, and assume that the sequence f_k converges to f pointwise almost everywhere. By multiplying f and each f_k by the characteristic function of some measurable set on which f_k converges pointwise to f, and which has complement of measure zero, we may take f_k to converge to f pointwise everywhere, and this only requires modifying f on a set of measure zero. In particular, the old f is measurable in the sense of b) if and only if the new f is.

The functions $\langle f_\ell \rangle_{\ell=1}^{\infty}$ clearly satisfy the condition b). Given an open set U of $\mathscr{H}$, define a sequence $U_1, U_2, \ldots$ of sets exhausting U as follows:

$$U_\ell = \left\{ x \in U \,\middle|\, U \supseteq \left\{ y \in \mathscr{H} \mid \|x - y\| < \tfrac{1}{2^\ell} \right\} \right\}$$

for each $\ell \in \mathbb{Z}_+$. Then

$$f^{-1}[U] = \bigcup_{k=1}^{\infty} \bigcup_{\ell=1}^{\infty} \bigcap_{m=\ell+1}^{\infty} f_m^{-1}[U_k]$$

is measurable.

b)$\Longrightarrow$c). Compositions of continuous functions with measurable ones are also measurable.

c)$\Longrightarrow$a). Fix an orthonormal basis $\langle\varphi_\ell\rangle_{\ell=1}^{\infty}$ of $\mathscr{H}$, and define

$$f_\ell(\cdot) \overset{\text{def}}{=} \langle\varphi_\ell|f(\cdot)\rangle : Q \longrightarrow \mathbb{C}$$

for each $\ell \in \mathbb{Z}_+$. Now the functions f_ℓ are measurable by the condition c) and they can be approximated from below (in the sense of absolute values) by simple functions; say $f_{\ell,m}\colon Q \longrightarrow \mathbb{C}$ ($m \in \mathbb{Z}_+$), $|f_{\ell,m}(\cdot)| \leqslant |f_\ell(\cdot)|$ on Q, and $f_{\ell,m}(x) \xrightarrow[m\longrightarrow\infty]{} f_\ell(x)$ for almost every $x \in Q$. Then

$$\left\langle \sum_{\ell=1}^{N} f_{\ell,N}(\cdot)\,\varphi_\ell \right\rangle_{N=1}^{\infty}$$

is a sequence of simple functions required by the condition a).

The Floquet transform $\mathscr{F}\!\ell$

We shall define the transform $\mathscr{F}\!\ell$ as the composition of four Hilbert space isomorphisms as follows:

$$\mathscr{F}\!\ell \overset{\text{def}}{=} \mathscr{W} \circ \mathscr{V} \circ \mathscr{U} \circ \mathscr{F},$$

where

$$\mathscr{L}^2(\mathbb{R}^n) \overset{\mathscr{F}}{\cong} \mathscr{L}^2(\mathbb{R}^n) \overset{\mathscr{U}}{\cong} \ell^2\big(\mathbb{Z}^n;\mathscr{L}^2(Q)\big) \overset{\mathscr{V}}{\cong} \mathscr{L}^2\big(Q;\ell^2(\mathbb{Z}^n)\big) \overset{\mathscr{W}}{\cong} \mathscr{L}^2\big(Q;\mathscr{L}^2(Q)\big).$$

Each of these is easy to describe. The first isomorphism, $\mathscr{F}$, is the usual Fourier transform.

The operator $\mathscr{U}$ splits each $f \in \mathscr{L}^2(\mathbb{R}^n)$ to a family of translations of restrictions $f\big|_{Q+\xi}(\cdot+\xi)$ ($\xi \in \mathbb{Z}^n$) of f. In other words,

$$\big(\mathscr{U}f\big)(\xi)(k) \overset{\text{def}}{=} f(\xi+k)$$

for every $\xi \in \mathbb{Z}^n$ and for almost every $k \in Q$. This is an isometry since for any $f \in \mathscr{L}^2(\mathbb{R}^n)$

$$\begin{aligned}\|\mathscr{U}f\|^2_{\ell^2(\mathbb{Z}^n;\mathscr{L}^2(Q))} &= \sum_{\xi\in\mathbb{Z}^n} \big\|(\mathscr{U}f)(\xi)\big\|^2_{\mathscr{L}^2(Q)} = \sum_{\xi\in\mathbb{Z}^n} \int_Q \big|(\mathscr{U}f)(\xi)(k)\big|^2\,\mathrm{d}k \\ &= \sum_{\xi\in\mathbb{Z}^n} \int_Q \big|f(\xi+k)\big|^2 \mathrm{d}k = \int_{\mathbb{R}^n} \big|f(x)\big|^2 \mathrm{d}x = \|f\|^2_{\mathscr{L}^2(\mathbb{R}^n)}.\end{aligned}$$

It is also clear that $\mathscr{U}$ is bijective; the inverse is given by the formula

$$\big(\mathscr{U}^{-1}g\big)(x) = g\big(\lfloor x_1\rfloor, \lfloor x_2\rfloor, \ldots, \lfloor x_n\rfloor\big)\big(\{x_1\}, \{x_2\}, \ldots, \{x_n\}\big)$$

for almost every $x \in \mathbb{R}^n$, and for every $g \in \ell^2\big(\mathbb{Z}^n;\mathscr{L}^2(Q)\big)$. Here $\{\alpha\}$ means the fractional part of a real number α, defined by the equation $\lfloor\alpha\rfloor + \{\alpha\} = \alpha$, as usual. Since the measure on $\mathbb{Z}^n$ is here the counting measure, problems due to measurability can not occur.

The operator $\mathscr{V}$ is even simpler. The elements of $\ell^2\big(\mathbb{Z}^n;\mathscr{L}^2(Q)\big)$ are ultimately just functions of two variables. The transform $\mathscr{V}$ exchanges these variables. More precisely, for any $f\in\ell^2\big(\mathbb{Z}^n;\mathscr{L}^2(Q)\big)$ we define the function $\mathscr{V}f\in\mathscr{L}^2\big(Q;\ell^2(\mathbb{Z}^n)\big)$ by the formula

$$\big(\mathscr{V}f\big)(k)(\xi)=f(\xi)(k)$$

for almost every $k\in Q$ and every $\xi\in\mathbb{Z}^n$. The isometry of $\mathscr{V}$ is established almost the same way as before: for any $f\in\ell^2\big(\mathbb{Z}^n;\mathscr{L}^2(Q)\big)$,

$$\begin{aligned}\big\|\mathscr{V}f\big\|^2_{\mathscr{L}^2(Q;\ell^2(\mathbb{Z}^n))}&=\int_Q\big\|\big(\mathscr{V}f\big)(k)\big\|^2_{\ell^2(\mathbb{Z}^n)}\mathrm{d}k=\int_Q\sum_{\xi\in\mathbb{Z}^n}\big|f(\xi)(k)\big|^2\mathrm{d}k\\&=\sum_{\xi\in\mathbb{Z}^n}\int_Q\big|f(\xi)(k)\big|^2\mathrm{d}k=\big\|f\big\|^2_{\ell^2(\mathbb{Z}^n;\mathscr{L}^2(Q))}.\end{aligned}$$

It is again clear that there are no measurability issues and that $\mathscr{V}$ is really a bijection.

We remark that we have now done enough work for the Thomas' argument to work. The main estimates are made in $\ell^2(\mathbb{Z}^n)$ and indeed, Thomas' original paper [Th] uses the transform $\mathscr{V}\circ\mathscr{U}\circ\mathscr{F}$.

The last link in the chain is the mapping $\mathscr{W}$ which regards each vector $\varphi\in\ell^2(\mathbb{Z}^n)$ as the Fourier coefficients of some function in $\mathscr{L}^2(Q)$. That is, for any $f\in\mathscr{L}^2\big(Q;\ell^2(\mathbb{Z}^n)\big)$, we define

$$\big(\mathscr{W}f\big)(k)(x)=\sum_{\xi\in\mathbb{Z}^n}f(k)(\xi)\,e(x\cdot\xi)$$

for almost every $k\in Q$ and $x\in Q$. Here, of course, the series is taken to converge in the $\mathscr{L}^2(Q)$-norm. The measurability of $\mathscr{W}f$ is again clear assuming that of f, and the isometry property of $\mathscr{W}$ follows from the elementary theory of Fourier series:

$$\begin{aligned}\big\|\mathscr{W}f\big\|^2_{\mathscr{L}^2(Q;\mathscr{L}^2(Q))}&=\int_Q\big\|\big(\mathscr{W}f\big)(k)\big\|^2_{\mathscr{L}^2(Q)}\mathrm{d}k=\int_Q\big\|f(k)\big\|^2_{\ell^2(\mathbb{Z}^n)}\mathrm{d}k\\&=\big\|f\big\|^2_{\mathscr{L}^2(Q;\ell^2(\mathbb{Z}^n))}.\end{aligned}$$

Finally, the inverse of $\mathscr{W}$ is given by simply transforming each fiber to its Fourier coefficients: for any $f\in\mathscr{L}^2\big(Q;\mathscr{L}^2(Q)\big)$,

$$\big(\mathscr{W}^{-1}f\big)(k)(\xi)=\int_Q f(k)(x)\,e(-x\cdot\xi)\,\mathrm{d}x$$

for almost every $k\in Q$ and for every $\xi\in\mathbb{Z}^n$.

We get the following expression for our Floquet transform $\mathscr{F}\ell$: for any f in $\mathscr{L}^2(\mathbb{R}^n)$,

$$\big(\mathscr{F}\ell\,f\big)(k)(x)=\sum_{\xi\in\mathbb{Z}^n}\widehat{f}(\xi+k)\,e(x\cdot\xi)\,,$$

for almost every $k\in Q$ and $x\in Q$, and where the series converges in the $\mathscr{L}^2$-norm. For, say Schwartz test functions $f\in\mathscr{S}(\mathbb{R}^n)$, the series converges also pointwise.

How does $\mathscr{F}\!\ell$ transform $\mathscr{H}^2(\mathbb{R}^n)$?

We now prove the first point mentioned in the beginning of the chapter:

Proposition. *For any $f \in \mathscr{H}^2(\mathbb{R}^n)$ we have $\big(\mathscr{F}\!\ell\, f\big)(k) \in \mathscr{H}^2(\mathbb{T}^n)$ for almost every $k \in Q$.*

By the very definition of the Sobolev space $\mathscr{H}^2(\mathbb{R}^n)$,

$$\mathscr{F}\big[\mathscr{H}^2(\mathbb{R}^n)\big] = \Big\{ f \in \mathscr{L}^2(\mathbb{R}^n) \,\Big|\, |\cdot|^2 f(\cdot) \in \mathscr{L}^2(\mathbb{R}^n) \Big\} .$$

Hence, in the same vein,

$$\begin{aligned}
\mathscr{V}\Big[\mathscr{U}\Big[\mathscr{F}\big[\mathscr{H}^2(\mathbb{R}^n)\big]\Big]\Big] &= \Big\{ f \in \mathscr{L}^2\big(Q; \ell^2(\mathbb{Z}^n)\big) \,\Big|\, |\cdot|^2 \big(\mathscr{U}^{-1}\mathscr{V}^{-1} f\big)(\cdot) \in \mathscr{L}^2(\mathbb{R}^n) \Big\} \\
&= \Big\{ f \in \mathscr{L}^2\big(Q; \ell^2(\mathbb{Z}^n)\big) \,\Big|\, (\xi + k)^2 f(k)(\xi) \in \ell^2\big(\mathbb{Z}^n; \mathscr{L}^2(Q)\big) \Big\} \\
&= \Big\{ f \in \mathscr{L}^2\big(Q; \ell^2(\mathbb{Z}^n)\big) \,\Big|\, \Big(1 + (\xi + k)^2\Big) f(k)(\xi) \in \ell^2\big(\mathbb{Z}^n; \mathscr{L}^2(Q)\big) \Big\} .
\end{aligned}$$

Here ξ and k denote the $\mathbb{Z}^n$- and Q-variables, respectively.

But now, since the variable k lives in the rather confined space Q, the expressions $1 + (\xi + k)^2$ and $1 + \xi^2$ are uniformly comparable. In fact,

$$\frac{1}{2 + 2n} \Big(1 + (\xi + k)^2\Big) \leqslant 1 + \xi^2 \leqslant (2 + 2n)\Big(1 + (\xi + k)^2\Big) .$$

Hence we have the characterisation

$$\mathscr{V}\Big[\mathscr{U}\Big[\mathscr{F}\big[\mathscr{H}^2(\mathbb{R}^n)\big]\Big]\Big] = \\
\Big\{ f \in \mathscr{L}^2\big(Q; \ell^2(\mathbb{Z}^n)\big) \,\Big|\, \big(1 + \xi^2\big) f(k)(\xi) \in \ell^2\big(\mathbb{Z}^n; \mathscr{L}^2(Q)\big) \Big\} ,$$

from which it is obvious that for any $f \in \mathscr{H}^2(\mathbb{R}^n)$,

$$\big(\mathscr{F}\!\ell\, f\big)(k) \in \mathscr{H}^2(\mathbb{T}^n)$$

for almost every $k \in Q$.

——:——

We take a quick break to mention the usual, non-modified, Floquet transform, which we denote by Fl. It is essentially given by the formulas

$$\big(\mathrm{Fl}\, f\big)(k)(x) = \sum_{\xi \in \mathbb{Z}^n} \widehat{f}(\xi + k)\, e\big(x \cdot (\xi + k)\big) = \sum_{\xi \in \mathbb{Z}^n} f(x + \xi)\, e(-\xi \cdot k)$$

for every $f \in \mathscr{L}^2(\mathbb{R}^n)$, understood in a suitable sense. It turns out that this gives rise to a Hilbert space isomorphism

$$\mathrm{Fl}\colon \mathscr{L}^2(\mathbb{R}^n) \cong \mathscr{L}^2\big(\mathbb{T}^n; \mathscr{L}^2(Q)\big) .$$

The primary reason why this transform is less suitable for our purposes than the modified version $\mathscr{Fl}$, has to do with the way Fl transforms $\mathscr{H}^2(\mathbb{R}^n)$-functions. Namely, even though

$$\big(\mathrm{Fl}\, f\big)(k) \in \mathscr{H}^2(Q)$$

for almost every $k \in \mathbb{T}^n$ and for every $f \in \mathscr{H}^2(\mathbb{R}^n)$, the fibre $\big(\mathrm{Fl}\, f\big)(k)$, instead of satisfying periodic boundary conditions, satisfies the more complicated condition

$$\big(\mathrm{Fl}\, f\big)(k)(\cdot + \xi) = e(k \cdot \xi)\, \big(\mathrm{Fl}\, f\big)(k)(\cdot)\,,$$

where $\xi \in \mathbb{Z}^n$ is arbitrary, and similar conditions for first derivatives.

How does $\mathscr{Fl}$ transform periodic Schrödinger operators?

We end the chapter by proving the second point mentioned at the beginning:

Proposition. *Let $W \in \mathscr{L}^\infty(\mathbb{R}^n;\mathbb{C}^n)$ and $V \in \mathscr{L}^\infty(\mathbb{R}^n)$ be periodic. Then for any $f \in \mathscr{H}^2(\mathbb{R}^n)$, we have*

$$\big(\mathscr{Fl}\, (-\Delta - iW\cdot\nabla + V)\, f\big)(k) = (\Delta_k + W\cdot\nabla_k + V)\, \big(\mathscr{Fl}\, f\big)(k)$$

for almost every $k \in Q$ where

$$\nabla_k = -i\nabla + 2\pi k, \quad \text{and} \quad \Delta_k = \nabla_k^2 = (-i\nabla + 2\pi k)^2\,.$$

We first compute $\mathscr{Fl} \circ (-i\partial_\ell)$, where $\ell \in \{1,2,\ldots,n\}$. Let $f \in \mathscr{H}^2(\mathbb{R}^n)$ be arbitrary. Since

$$\big(\mathscr{F}\,(-i\partial_\ell f)\big)(x) = 2\pi x_\ell \widehat{f}(x)$$

for almost every $x \in \mathbb{R}^n$, we have

$$\big(\mathscr{V}\mathscr{U}\mathscr{F}\,(-i\partial_\ell f)\big)(k)(\xi) = 2\pi\,(\xi_\ell + k_\ell)\,\widehat{f}(\xi + k)$$

for every $\xi \in \mathbb{Z}^n$ and for almost every $k \in Q$. Finally, since

$$\begin{aligned}\big(\mathscr{W}\mathscr{V}\mathscr{U}\mathscr{F}\,(-i\partial_\ell f)\big)(k)(x) &= 2\pi \sum_{\xi\in\mathbb{Z}^n} (\xi_\ell + k_\ell)\,\widehat{f}(\xi+k)\,e(x\cdot\xi) \\ &= (-i\partial_\ell + 2\pi k_\ell) \sum_{\xi\in\mathbb{Z}^n} \widehat{f}(\xi+k)\,e(x\cdot\xi)\,,\end{aligned}$$

we conclude that

$$\big(\mathscr{Fl}\,(-i\partial_\ell f)\big)(k) = (-i\partial_\ell + 2\pi k_\ell)\,\big(\mathscr{Fl}\, f\big)(k) \qquad (\alpha)$$

for almost every $k \in Q$.

Next, let $V \in \mathscr{L}^\infty(\mathbb{R}^n)$ be periodic. Since the multiplication operator $\mathscr{M}_V$ is bounded, we may restrict our attention to computing $\mathscr{Fl}\,(Vf)$ for, say, f in $\mathscr{S}(\mathbb{R}^n)$. Let us write V as a Fourier series

$$V(x) = \sum_{\xi\in\mathbb{Z}^n} \widehat{V}_\xi\, e(x\cdot\xi)$$

converging in the $\mathscr{L}^2$-norm in Q and in any of its translates. Now

$$\big(\mathscr{F}\,(Vf)\big)(x) = \sum_{\xi\in\mathbb{Z}^n} \widehat{V}_\xi\, \widehat{e(\xi\cdot\cdot)\,f(\cdot)}(x) = \sum_{\xi\in\mathbb{Z}^n} \widehat{V}_\xi\, \widehat{f}(x-\xi)\,,$$

and consequently

$$\big(\mathscr{V}\mathscr{U}\mathscr{F}\,(Vf)\big)(k)(\xi) = \sum_{\eta\in\mathbb{Z}^n} \widehat{V}_\eta\, \widehat{f}(\xi+k-\eta)\,,$$

for almost every $k \in Q$ and for every $\xi \in \mathbb{Z}^n$. The final step is simply to see that for almost every $k \in Q$ and $x \in Q$,

$$\begin{aligned}
\big(\mathscr{F\!\ell}\,(Vf)\big)(k)(x) &= \sum_{\xi\in\mathbb{Z}^n}\sum_{\eta\in\mathbb{Z}^n} \widehat{V}_\eta\, \widehat{f}(\xi+k-\eta)\, e(x\cdot\xi) \\
&= \sum_{\zeta\in\mathbb{Z}^n}\sum_{\eta\in\mathbb{Z}^n} \widehat{V}_\eta\, e(\eta\cdot x)\, \widehat{f}(\zeta+k)\, e(\zeta\cdot x) \\
&= V(x) \sum_{\zeta\in\mathbb{Z}^n} \widehat{f}(\zeta+k)\, e(\zeta\cdot x) \\
&= V(x)\,\big(\mathscr{F\!\ell}\,f\big)(k)(x)\,,
\end{aligned}$$

so that for every $f \in \mathscr{L}^2(\mathbb{R}^n)$ and almost every $k \in Q$,

$$\big(\mathscr{F\!\ell}\,(Vf)\big)(k) = V\,\big(\mathscr{F\!\ell}\,f\big)(k)\,. \qquad (\beta)$$

Finally, let $W \in \mathscr{L}^\infty(\mathbb{R}^n;\mathbb{C}^n)$ and $V \in \mathscr{L}^\infty(\mathbb{R}^n)$ be periodic. Combining the formulas (α) and (β), and defining

$$\nabla_k \stackrel{\text{def}}{=} -i\nabla + 2\pi k, \quad \text{and} \quad \Delta_k \stackrel{\text{def}}{=} (-i\nabla + 2\pi k)^2\,,$$

gives

$$\begin{aligned}
&\big(\mathscr{F\!\ell}\,(-\Delta f - iW\cdot\nabla f + Vf)\big)(k) \\
&\qquad = \Big((-i\nabla+2\pi k)^2 + W\cdot(-i\nabla+2\pi k) + V\Big)\big(\mathscr{F\!\ell}\,f\big)(k) \\
&\qquad = (\Delta_k + W\cdot\nabla_k + V)\big(\mathscr{F\!\ell}\,f\big)(k)
\end{aligned}$$

for every $f \in \mathscr{H}^2(\mathbb{R}^n)$ and almost every $k \in Q$.

Analytic Families of Type $\mathscr{A}$ and the Analytic Fredholm Theorem

In this chapter we shall discuss analytic families of operators. The notion of analytic function generalizes effortlessly to Banach space valued, and hence also bounded operator valued, functions. We also need a concept for analytic families of unbounded operators. This is provided by analytic families of operators of type $\mathscr{A}$, the main examples of which will be Floquet transforms of periodic Schrödinger operators. This is rather natural considering that the Floquet transforms in question are in fact second degree polynomials in k.

The main point of this chapter is the analytic Fredholm theorem. The only result of this chapter needed in later discussions is that the conclusion of the analytic Fredholm theorem applies to the Floquet transforms of the operators we consider.

For a discussion on basic function theory on the level we need, we refer to chapter 10 of Shilov's textbook [Shi1], and for the corresponding Banach space valued theory, we refer to the section 1.6 of the second volume [Shi2]. Our discussion of analytic families of type $\mathscr{A}$ is based on chapter 15 of P. D. Hislop's and I. M. Sigal's book [Hi&S]. The classic text [Ka] of T. Kato and the section XII.2 of M. Reed's and B. Simon's volume [R&S2] were also very useful. Our proof of the analytic Fredholm theorem follows closely that of theorem VI.14 of Reed and Simon's book [R&S1, VI.5].

Analytic Banach space -valued functions

Let Ω be a non-empty open subset of the complex plane and let $\mathscr{X}$ be a Banach space whose norm we denote by $\|\cdot\|$. We call a function $\varphi\colon \Omega \longrightarrow \mathscr{X}$ **analytic**, if for every $z \in \Omega$, the limit

$$\lim_{\Delta z \longrightarrow 0} \frac{\varphi(z+\Delta z) - \varphi(z)}{\Delta z}$$

exists in $\mathscr{X}$. When this is the case, we denote the above limit by $\varphi'(z)$, thereby obtaining another function $\varphi'\colon \Omega \longrightarrow \mathscr{X}$, the **derivative** of φ. We shall be flexible with terminology, and instead of analytic, we sometimes call φ a **vector-valued** or a **Banach space valued** analytic function. In the case in which $\mathscr{X}$ is the Banach space of bounded linear operators of some Hilbert space, equipped with the usual operator norm, we also call φ an **operator-valued** analytic function.

A typical example of an operator-valued function is furnished by the resolvent operators $R_A(\cdot)$ of some closed linear operator A, with non-empty resolvent set, in some Hilbert space.

It is an important fact concerning such Banach space -valued analytic functions that the basic theory of ordinary analytic functions immediately generalizes to them simply by changing some absolute value signs $|\cdot|$ to Banach space norms $\|\cdot\|$. For example, vector-valued analytic functions are continuous so that the usual Riemann line integrals generalize without any problems, and we know that vector-valued analytic functions are infinitely differentiable and locally expressible as uniformly convergent Taylor series. For the above analytic function φ, and for any disc $\overline{B}(z,\varepsilon) = \{\zeta \in \Omega \mid |z-\zeta| \leqslant \varepsilon\}$, where $z \in \Omega$ and $\varepsilon \in \mathbb{R}_+$, contained in Ω, we have the usual Cauchy integral formula:

$$\varphi(z) = \frac{1}{2\pi i} \oint_{\partial B(z,\varepsilon)} \frac{\varphi(\zeta)\,\mathrm{d}\zeta}{\zeta - z},$$

where the boundary of the disc is oriented in the usual way.

——:——

We introduce another concept of analyticity for operator-valued functions. Let $\mathscr{H}$ be a separable infinite-dimensional Hilbert space with norm $\|\cdot\|$, and let $A\colon \Omega \longrightarrow \mathscr{B}(\mathscr{H})$. We say that A is **strongly analytic**, if the vector-valued function

$$z \longmapsto A(z)\,\varphi\colon \Omega \longrightarrow \mathscr{H}$$

is analytic for every fixed vector $\varphi \in \mathscr{H}$. The punch line is that this new notion is superfluous:

Proposition. *All strongly analytic operator-valued functions are analytic.*

Let $A\colon \Omega \longrightarrow \mathscr{B}(\mathscr{H})$ be strongly analytic, and let $\Gamma \subseteq \Omega$ be compact. Since the mapping

$$z \longmapsto \big\|A(z)\,\varphi\big\|\colon \Gamma \longrightarrow [0,\infty[$$

is continuous, and therefore bounded, for every $\varphi \in \mathscr{H}$, the Banach–Steinhaus theorem guarantees that the function

$$z \longmapsto \big\|A(z)\big\|\colon \Gamma \longrightarrow [0,\infty[$$

is bounded as well.

But now for every disc $\overline{B}(z_0,\varepsilon) \subseteq \Omega$ ($z_0 \in \Omega$, $\varepsilon \in \mathbb{R}_+$) we can define an analytic operator-valued function $B\colon B(z_0,\varepsilon) \longrightarrow \mathscr{B}(\mathscr{H})$ via the formula

$$B(z) = \frac{1}{2\pi i} \oint_{\partial B(z_0,\varepsilon)} \frac{A(\zeta)\,\mathrm{d}\zeta}{\zeta - z},$$

which is to hold for all $z \in B(z_0, \varepsilon)$. On the other hand, the Cauchy integral theorem tells us that

$$A(z)\,\varphi = \frac{1}{2\pi i} \oint\limits_{\partial B(z_0,\varepsilon)} \frac{A(\zeta)\,\varphi\,\mathrm{d}\zeta}{\zeta - z} = B(z)\,\varphi$$

for every $\varphi \in \mathscr{H}$ and each $z \in B(z_0, \varepsilon)$, thereby establishing the equality $A\big|_{B(z_0,\varepsilon)} = B$ and the analyticity of A.

Analytic families of type $\mathscr{A}$

Let $\Omega \subseteq \mathbb{C}$ be an open non-empty set, and let $T = z \longmapsto T_z \colon \Omega \longrightarrow \mathscr{C}(\mathscr{H})$ be a family of closed operators of some separable infinite-dimensional Hilbert space $\mathscr{H}$. We call T an **analytic family of type $\mathscr{A}$**, if

1. none of the resolvent sets $\rho(T_z)$, $z \in \Omega$, is empty;
2. the domain of T_z, $z \in \Omega$, is independent of z, in which case we denote the common domain by $\operatorname{Dom} T$; and
3. the mapping
$$z \longmapsto T_z\varphi \colon \Omega \longrightarrow \mathscr{H}$$
is analytic for every $\varphi \in \operatorname{Dom} T$.

For the rest of this section, we will assume that the family T introduced above is an analytic family of type $\mathscr{A}$. We shall prove three basic properties of analytic families of type $\mathscr{A}$. The first one we call "the upper semicontinuity of the spectrum", following [Ka, IV.§3.1]. The name is a reasonable one for the result essentially says that the spectrum of T_z can not instantaneously "explode"; that is, the spectrum of T_z can grow at most continuously with respect to z. This is analogous with the ordinary notion of upper semicontinuity of functions.

Upper semicontinuity of the spectrum. *Let $z_0 \in \Omega$ be an arbitrary fixed point and let $\mu \in \rho(T_{z_0})$. Then $\mu \in \rho(T_z)$ for z in Ω sufficiently close to z_0.*

The bounded resolvent operator $(\mu - T_{z_0})^{-1}$ is defined on all of $\mathscr{H}$ and has as its image the domain $\operatorname{Dom} T$. On the other hand, for every $z \in \Omega$, the closed operator $T_z - T_{z_0}$ has $\operatorname{Dom} T$ as its domain. The composed operators $(T_z - T_{z_0})(\mu - T_{z_0})^{-1}$ are then also closed, and being defined on all of $\mathscr{H}$, the closed graph theorem tells us that they are all bounded. But now we have obtained a strongly analytic family of bounded operators

$$z \longmapsto (T_z - T_{z_0})(\mu - T_{z_0})^{-1} : \Omega \longrightarrow \mathscr{B}(\mathscr{H})\,,$$

and by the considerations of the previous section this family must be analytic as well. Thus

$$(T_z - T_{z_0})(\mu - T_{z_0})^{-1} \xrightarrow[z \longrightarrow z_0]{} 0.$$

Obviously

$$\mu - T_z = \Big(1 - (T_z - T_{z_0})(\mu - T_{z_0})^{-1}\Big)(\mu - T_{z_0})$$

for every $z \in \Omega$. For z sufficiently near to z_0,

$$\left\|(T_z - T_{z_0})(\mu - T_{z_0})^{-1}\right\| < \frac{1}{2},$$

so that the operator $1 - (T_z - T_{z_0})(\mu - T_{z_0})^{-1}$ is invertible,

$$(\mu - T_z)^{-1} = (\mu - T_{z_0})^{-1}\Big(1 - \,(T_z - T_{z_0})(\mu - T_{z_0})^{-1}\Big)^{-1},$$

and $\mu \in \rho(T_z)$. q.e.d.

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The second property we prove just says that the resolvent operators of T_z depend analytically on z where defined. More precisely, continuing to use the notation introduced above, we get:

Proposition. *For any $\mu \in \rho(T_{z_0})$, the resolvent operator $(\mu - T_z)^{-1}$ exists and depends analytically on z for z near z_0.*

The proof is easy: for z so close to z_0 that

$$\left\|(T_z - T_{z_0})(\mu - T_{z_0})^{-1}\right\| < \frac{1}{2},$$

the series in the identity

$$(\mu - T_z)^{-1} = (\mu - T_{z_0})^{-1}\sum_{\ell=0}^{\infty}\Big((T_z - T_{z_0})\,(\mu - T_{z_0})^{-1}\Big)^{\ell}$$

is a uniformly convergent series of analytic operator-valued functions, and therefore analytic itself.

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From the above considerations we get easily another property of type $\mathscr{A}$ analytic families:

Proposition. *If Ω is connected, then **either** T_z has compact resolvent for all $z \in \Omega$ **or** T_z has compact resolvent for no $z \in \Omega$.*

The point is that the set ω of points of Ω at which T has compact resolvent is clopen in Ω. It is open since for any $z_0 \in \omega$ and for any $\mu \in \rho(T_{z_0})$, we have that for z sufficiently near to z_0, $\mu \in \rho(T_z)$ and the resolvent operator

$$(\mu - T_z)^{-1} = (\mu - T_{z_0})^{-1}\left(1 - (T_z - T_{z_0})(\mu - T_{z_0})^{-1}\right)^{-1}$$

is compact since the operator $(\mu - T_{z_0})^{-1}$ is.

On the other hand, the set ω is closed since for any accumulation point $z_0 \in \Omega$ of ω, the resolvent operators of T_{z_0} may be approximated in the operator norm by the compact resolvent operators of T_z with $z \in \omega$.

The analytic Fredholm theorem

In what follows, $\mathscr{H}$ continues to be a separable infinite-dimensional Hilbert space. The purpose of this section is to prove that an analytic family of compact operators, or an analytic family of type $\mathscr{A}$ of closed operators with compact resolvents, either has a given complex number as an eigenvalue everywhere or merely in a discrete set of the values of the parameter. We begin by proving this for the former after which the result easily follows for the latter.

The analytic Fredholm theorem. *Let $\Omega \subseteq \mathbb{C}$ be non-empty, open and connected, let $A = z \longmapsto A_z\colon \Omega \longrightarrow \mathscr{B}(\mathscr{H})$ be an analytic operator-valued function taking only compact values, and let α be an arbitrary non-zero complex number. Then* ***either***

> *α is an eigenvalue of A_z for all $z \in \Omega$,*

or

> *the set of points $z \in \Omega$, for which α is an eigenvalue of A_z, is discrete.*

Let $z_0 \in \Omega$ be arbitrary. By the connectedness of Ω it clearly suffices to prove that in a neighbourhood ω of z_0 in Ω, either α is an eigenvalue of A_z for every $z \in \omega$ or the set of points $z \in \omega$, for which α is an eigenvalue of A_z, is discrete. The reason for this is that once this local formulation has been proved, Ω becomes immediately the union of two manifestly open sets: the set of points in the neighbourhoods in which the first possibility always occurs, and the set of points in the neighbourhoods in which the latter situation reigns. By connectedness of Ω one of these two sets must be empty.

Let $\varepsilon \in \mathbb{R}_+$ be so small that $\overline{B}(z_0, \varepsilon) \subseteq \Omega$ and that

$$\left\| A_z - A_{z_0} \right\| < \frac{|\alpha|}{4}$$

for all $z \in B(z_0, \varepsilon)$. Since in Hilbert spaces compact operators may be approximated arbitrarily well by finite-rank operators, there must exist a bounded finite-rank operator F of $\mathscr{H}$ for which

$$\left\| A_{z_0} - F \right\| < \frac{|\alpha|}{4}.$$

Now $\left\| A_z - F \right\| < \frac{|\alpha|}{2}$ for every $z \in B(z_0, \varepsilon)$ and hence the Neumann series

$$(\alpha - A_z + F)^{-1} = \frac{1}{\alpha} \sum_{\ell=0}^{\infty} \frac{1}{\alpha^\ell} (A_z - F)^\ell$$

converges uniformly in $\mathscr{B}(\mathscr{H})$ for $z \in B(z_0, \varepsilon)$. The point of this observation is that the operator $(\alpha - A_z + F)^{-1}$ exists for each $z \in B(z_0, \varepsilon)$ and depends analytically on z.

Next, define a new analytic family of bounded finite-rank operators

$$B = z \longmapsto B_z\colon B(z_0, \varepsilon) \longrightarrow \mathscr{B}(\mathscr{H})$$

by setting $B_z = F\,(\alpha - A_z + F)^{-1}$. Since

$$\alpha - A_z = (1 - B_z)(\alpha - A_z + F)$$

for each $z \in B(z_0, \varepsilon)$, the operator $\alpha - A_z$ is invertible precisely when the operator $1 - B_z$ is. Furthermore, we have $A_z\varphi = \alpha\varphi$ for some vector $\varphi \in \mathscr{H} \setminus \{0\}$ if and only if $B_z\varphi = \varphi$ for some $\varphi \in \mathscr{H} \setminus \{0\}$. But the operator B_z is a finite-rank operator for each value of z. Hence, by introducing F, we have reduced the problem to a finite-dimensional one. The rest of the proof revolves around capitalizing on this finite-dimensionality.

Let $\psi_1, \psi_2, \ldots, \psi_n$ ($n \in \mathbb{Z}_+$) be an orthonormal basis of $\operatorname{Im} F \subsetneq \mathscr{H}$. Then $F = \sum_{\ell=1}^{n} \alpha_\ell(\cdot)\,\psi_\ell$ for some continuous linear functionals $\alpha_1, \alpha_2, \ldots, \alpha_n$ of $\mathscr{H}$, and by the Riesz representation theorem, there exists unique vectors $\varphi_1, \varphi_2, \ldots, \varphi_n \in \mathscr{H}$ such that

$$F = \sum_{\ell=1}^{n} \langle \varphi_\ell \,|\, \cdot \rangle\, \psi_\ell.$$

Since $\operatorname{Im} B_z \subseteq \operatorname{Im} F$ for every $z \in B(z_0, \varepsilon)$ and B_z depends analytically on z, there must exist analytic functions

$$\varphi_1(\cdot)\,, \varphi_2(\cdot)\,, \ldots, \varphi_n(\cdot) \colon B(z_0, \varepsilon) \longrightarrow \mathscr{H}$$

for which

$$B_z = \sum_{\ell=1}^{n} \langle \varphi_\ell(z) | \cdot \rangle\, \psi_\ell$$

for each and every $z \in B(z_0, \varepsilon)$.

Let us consider a fixed point $\psi \in \mathscr{H}$ of B_z for some temporarily fixed $z \in B(z_0, \varepsilon)$. Since $\psi \in \operatorname{Im} F$, there exist unique coefficients $\beta_1, \beta_2, \ldots, \beta_n \in \mathbb{C}$ such that

$$\psi = \sum_{\ell=1}^{n} \beta_\ell \psi_\ell.$$

Since for each $k \in \{1, 2, \ldots, n\}$, we have

$$\begin{aligned}\beta_k = \langle \psi_k \,|\, \psi \rangle = \langle \psi_k \,|\, B_z\psi \rangle &= \Big\langle \psi_k \,\Big|\, \sum_{\ell=1}^{n} \langle \varphi_\ell(z) |\, \psi \rangle\, \psi_\ell \Big\rangle \\ = \langle \varphi_k(z) |\, \psi \rangle = \Big\langle \varphi_k(z) \Big|\, \sum_{\ell=1}^{n} \beta_\ell \psi_\ell \Big\rangle &= \sum_{\ell=1}^{n} \langle \varphi_k(z) |\, \psi_\ell \rangle\, \beta_\ell,\end{aligned}$$

the equation $B_z\psi = \psi$ is equivalent to the system of n linear equations

$$\begin{cases} \beta_1 = \sum_{\ell=1}^{n} \langle \varphi_1(z) |\, \psi_\ell \rangle\, \beta_\ell, \\ \beta_2 = \sum_{\ell=1}^{n} \langle \varphi_2(z) |\, \psi_\ell \rangle\, \beta_\ell, \\ \qquad \cdots\cdots\cdots\cdots\cdots \\ \beta_n = \sum_{\ell=1}^{n} \langle \varphi_n(z) |\, \psi_\ell \rangle\, \beta_\ell. \end{cases}$$

These, in turn, may be written in the even simpler form $M(z)\,\beta = 0$, where $\beta = \langle \beta_1, \beta_2, \ldots, \beta_n \rangle \in \mathbb{C}^n$ and $M(z)$ is the complex $n \times n$-matrix

$$\begin{bmatrix} \langle \varphi_1(z) |\, \psi_1 \rangle - 1 & \langle \varphi_1(z) |\, \psi_2 \rangle & \cdots & \langle \varphi_1(z) |\, \psi_n \rangle \\ \langle \varphi_2(z) |\, \psi_1 \rangle & \langle \varphi_2(z) |\, \psi_2 \rangle - 1 & \cdots & \langle \varphi_2(z) |\, \psi_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \varphi_n(z) |\, \psi_1 \rangle & \langle \varphi_n(z) |\, \psi_2 \rangle & \cdots & \langle \varphi_n(z) |\, \psi_n \rangle - 1 \end{bmatrix}.$$

Therefore α is an eigenvalue of A_z if and only if the equation $B_z\psi = \psi$ has a non-zero solution in $\mathscr{H}$, and this happens precisely when $\det M(z) = 0$. But the function $\det M(\cdot)$ is clearly analytic in $B(z_0, \varepsilon)$, and thus either it vanishes identically in $B(z_0, \varepsilon)$ or its set of zeros is a discrete subset of $B(z_0, \varepsilon)$, and so we are done.

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We will actually apply the following variant of this result, and in the lack of a better name we call it with the same one.

The analytic Fredholm theorem. *Let again $\Omega \subseteq \mathbb{C}$ be non-empty, open and connected, let $T = z \longmapsto T_z \colon \Omega \longrightarrow \mathscr{C}(\mathscr{H})$ be an analytic family of type $\mathscr{A}$, suppose that T_z has compact resolvent for each $z \in \Omega$, and let $\lambda \in \mathbb{C}$ be arbitrary. Then* ***either***

λ is an eigenvalue of T_z for every $z \in \Omega$,

or

the set of points $z \in \Omega$, for which λ is an eigenvalue of T_z, is discrete.

Let $z_0 \in \Omega$ be arbitrary. It is again enough to show that in a neighbourhood ω of z_0 in Ω, either λ is an eigenvalue of T_z for all $z \in \omega$ or the set of points $z \in \omega$, for which λ is an eigenvalue of T_z, is discrete.

Let $\zeta_0 \in \rho(T_{z_0}) \setminus \{\lambda\}$ be arbitrary, and choose a neighbourhood ω of z_0 in Ω so that $\zeta_0 \in \rho(T_z)$ for every $z \in \omega$. Then the operator-valued function

$$z \longmapsto (\zeta_0 - T_z)^{-1} \colon \omega \longrightarrow \mathscr{B}(\mathscr{H})$$

obtains only compact values and is analytic. Now it is clear that $T_z\varphi = \lambda\varphi$ for some non-zero vector $\varphi \in \operatorname{Dom} T$ if and only if

$$(\zeta_0 - T_z)^{-1}\varphi = \frac{1}{\zeta_0 - \lambda}\varphi$$

for some non-zero vector $\varphi \in \mathscr{H}$. Thus the result follows from the previous analytic Fredholm theorem applied with $\alpha = \frac{1}{\zeta_0 - \lambda}$.

Floquet decompositions as analytic families

As advertised before, the main point of this chapter is that Floquet decompositions of periodic magnetic Schrödinger operators give rise to analytic families of type $\mathscr{A}$ to which analytic Fredholm theorem may then be applied. To be more precise, there are two things to prove here. In this section we will be concerned with the first one:

Proposition. *Let $V \in \mathscr{L}^\infty(\mathbb{R}^n)$ and $W \in \mathscr{L}^\infty(\mathbb{R}^n; \mathbb{C}^n)$ be periodic. Then the family of operators*

$$k \longmapsto \Delta_k + W \cdot \nabla_k + V,$$

defined for every $k \in \mathbb{C}^n$, and each operator having domain $\mathscr{H}^2(\mathbb{T}^n)$, is an analytic family of type $\mathscr{A}$ in each of the components of k.

The second observation, the proof of which is relegated to the beginning of the next chapter, encapsulates the actual way in which we will actually use the analytic Fredholm theorem. Note that the analytic Fredholm theorem is a single variable result whereas our families of interest are multivariable. One could also use the more abstract theory of analytic Fredholm operator -valued functions of several variables — for more details, we refer to Kuchment's book [Ku1].

Proposition. *Let us be given an arbitrary $\lambda \in \mathbb{C}$ such that λ is an eigenvalue of $\Delta_k + W \cdot \nabla_k + V \colon \mathscr{H}^2(\mathbb{T}^n) \longrightarrow \mathscr{L}^2(\mathbb{T}^n)$ for k in a subset of $\mathbb{R}^n$ of strictly positive measure. Then λ is an eigenvalue of $\Delta_k + W \cdot \nabla_k + V$ for every $k \in \mathbb{C}^n$.*

The proof of the first proposition naturally entails two things:

1. We have to show that the operator
$$\Delta_k + W \cdot \nabla_k + V \colon \mathscr{H}^2(\mathbb{T}^n) \longrightarrow \mathscr{L}^2(\mathbb{T}^n)$$
is indeed a closed operator of $\mathscr{L}^2(\mathbb{T}^n)$ with non-empty resolvent set for every $k \in \mathbb{C}^n$.
2. We also have to show that the vector-valued function
$$k \longmapsto \Delta_k \varphi + W \cdot \nabla_k \varphi + V\varphi \colon \mathbb{C}^n \longrightarrow \mathscr{L}^2(\mathbb{T}^n)$$
is analytic for each $\varphi \in \mathscr{H}^2(\mathbb{T}^n)$.

Of these, the second one is trivial since the vector-valued function in question is a second-degree polynomial in each of the components of k, and the analyticity is obtained with the simple high-school argument.

To prove the first one, it suffices to establish the non-emptiness of the resolvent set of $\Delta_k + W \cdot \nabla_k + V$. So, let us fix the value of $k \in \mathbb{C}^n$ for the rest of this section. Our goal is to show that $\mu \in \rho(\Delta_k + W \cdot \nabla_k + V)$ when μ is a sufficiently large negative real number. The key point is the observation:

Observation. *For each $\mu \in \left]-\infty, -4\pi^2 (\Im k)^2\right[$, we have $\mu \in \rho(\Delta_k)$ and*
$$\left\|(\mu - \Delta_k)^{-1}\right\| \ll \frac{1}{\left|\mu + 4\pi^2 (\Im k)^2\right|}.$$

The proof is simple: On the Fourier series side the operator Δ_k is just multiplication by the polynomial
$$4\pi^2 (\xi + k)^2 = 4\pi^2 (\xi + \Re k)^2 - 4\pi^2 (\Im k)^2 + 8\pi^2 i (\xi + \Re k) \cdot \Im k.$$

In particular the real part, which consists of the first two terms of the right-hand side, is at least $-4\pi^2 (\Im k)^2$, and so for $\mu \in \left]-\infty, -4\pi^2 (\Im k)^2\right[$, the resolvent operator $(\mu - \Delta_k)^{-1}$ exists and is just division by the expression $\left(\mu - 4\pi^2 (\xi + k)^2\right)^{-1}$ on the Fourier series side, and certainly now
$$\left|\frac{1}{\mu - 4\pi^2 (\xi + k)^2}\right| \leqslant \frac{1}{\left|\mu + 4\pi^2 (\Im k)^2 - 4\pi^2 (\xi + \Re k)^2\right|} \leqslant \frac{1}{\left|\mu + 4\pi^2 (\Im k)^2\right|}.$$

It is also clear that the resolvent is compact for fixed values of k. In particular, we now have that

$$\left\|(\mu-\Delta_k)^{-1}\right\| \xrightarrow[\mu\longrightarrow-\infty]{} 0,$$

and since

$$\left|\frac{2\pi\left(\xi_\ell+k_\ell\right)}{\mu-4\pi^2\left(\xi+k\right)^2}\right| \longrightarrow 0$$

uniformly as $\mu \longrightarrow -\infty$, for each $\ell \in \{1,2,\ldots,n\}$, we easily conclude that

$$\left\|W\cdot\nabla_k\left(\mu-\Delta_k\right)^{-1}\right\| \xrightarrow[\mu\longrightarrow-\infty]{} 0.$$

But now

$$\mu-(\Delta_k+W\cdot\nabla_k+V)=\left(1-(W\cdot\nabla_k+V)(\mu-\Delta_k)^{-1}\right)(\mu-\Delta_k)$$

where the norm $\left\|(W\cdot\nabla_k+V)(\mu-\Delta_k)^{-1}\right\| \xrightarrow[\mu\longrightarrow-\infty]{} 0$, and thus for sufficiently large negative μ, everything on the right-hand side is invertible, and we have

$$\left(\mu-(\Delta_k+W\cdot\nabla_k+V)\right)^{-1}=(\mu-\Delta_k)^{-1}\left(1-(W\cdot\nabla_k+V)(\mu-\Delta_k)^{-1}\right)^{-1},$$

and we conclude that $\mu\in\rho(\Delta_k+W\cdot\nabla_k+V)$.

The Argument of Thomas

In this chapter we will present the approach of L. E. Thomas for proving absence of eigenvalues for periodic operators. The argument first appeared in Thomas' article [Th], and essentially all subsequent periodic absence of eigenvalues results rely on it.

We begin by proving absence of eigenvalues for $-\Delta + V$ with bounded periodic potential V. We also show how to handle periodicity with respect to a general lattice by showing absence of eigenvalues for $-\nabla \cdot G\nabla + V$ with bounded periodic potential V and a constant positive-definite symmetric real matrix G. We also give the simple algebraic trick that takes care of the situation where the matrix G is multiplied by a periodic positive scalar function.

In this chapter we briefly consider the problem of first-order terms, i.e. proving absence of eigenvalues for the periodic magnetic operators $(-i\nabla + A)^2 + V$. The main point is that even though Thomas' argument alone can not accommodate general magnetic potentials A, it easily manages sufficiently small ones. The connection to Sobolev's argument is that Sobolev's approach is ultimately about making a sizeable chunk of the magnetic potential to vanish by conjugating by carefully constructed toroidal pseudodifferential operators, leaving only a small magnetic potential which can then be easily dealt with.

Thomas' argument

We begin by combining the tools introduced in the previous chapters in order to prove, using as simple form of Thomas' argument as possible, that

> *for any periodic electric potential $V \in \mathscr{L}^\infty(\mathbb{R}^n)$, the Schrödinger operator $-\Delta + V$ with domain $\mathscr{H}^2(\mathbb{R}^n)$ has no eigenvalues.*

We shall argue by reductio ad absurdum and therefore assume that the operator in question does have an eigenvalue. By subtracting the eigenvalue from V, if necessary, we may assume that the eigenvalue is equal to zero.

The Floquet transform and its consequences

Let $f \in \mathscr{H}^2(\mathbb{R}^n)$ be such that $-\Delta f + Vf = 0$, even though $f \not\equiv 0$. This means that

$$\big(\mathscr{F}\!\ell\, f\big)(k) \not\equiv 0$$

for k in some subset Q' of Q of strictly positive measure, whereas

$$(\Delta_k + V)\big(\mathscr{F}\!\ell\, f\big)(k) = \Big((-i\nabla + 2\pi k)^2 + V\Big)\big(\mathscr{F}\!\ell\, f\big)(k) \equiv 0$$

for almost every $k \in Q$.

We will momentarily consider $\Delta_k + V$ to be an analytic family of type $\mathscr{A}$ with respect to each of the variables $k_1, k_2, \ldots, k_n$. As mentioned before, the polynomial nature of $\Delta_k + V$ allows us to extend it to an entire analytic family of type $\mathscr{A}$ with respect to each of the variables $k_1, k_2, \ldots, k_n$. We want argue via the analytic Fredholm theorem, that zero is an eigenvalue of $\Delta_k + V$ for every $k \in \mathbb{C}^n$.

This is achieved by a simple finite induction. For the purposes of this section, we write $I = [0, 1[$. First, let Q_2 be the set of elements $\langle x_2, x_3, \ldots, x_n \rangle \in I^{n-1}$ for which the set of $x_1 \in I$ for which $\langle x_1, x_2, \ldots, x_n \rangle$ belongs to Q' is of positive measure. Then the analytic Fredholm theorem guarantees that for each $k' \in Q_2$, zero must be an eigenvalue of $\Delta_k + V$ for each $k \in \mathbb{C} \times \{k'\}$, since sets of positive measure can not be discrete. Thus, zero is an eigenvalue of $\Delta_k + V$ for $k \in \mathbb{C} \times Q_2$.

But now, simply fix the value of k_1, and precisely the same argument, rewritten in $(n-1)$-dimensional form, proves that zero must be an eigenvalue of $\Delta_k + V$ for every $k \in \mathbb{C}^2 \times Q_3$, where Q_3 is the set of elements $\langle x_3, x_4, \ldots, x_n \rangle \in I^{n-2}$ such that the set of $x_2 \in I$ for which zero is an eigenvalue of $\Delta_k + V$ with $k = \langle k_1, x_2, \ldots, x_n \rangle$ is of positive measure (in $\mathbb{R}$).

After $n - 2$ more steps, we have proved that zero is an eigenvalue of $\Delta_k + V$ for every $k \in \mathbb{C}^n$. For further reference, we remark that the precise form of the operator $\Delta_k + V$ was irrelevant in the proof, so that the same conclusion applies also to operators of the more general form $\nabla_k \cdot G \nabla_k + W \cdot \nabla_k + V$ which will be considered later.

The main estimate

We have proved that $\Delta_k + V$ has zero as its eigenvalue for every $k \in \mathbb{C}^n$. Thus, to reach a contradiction, it suffices that we produce a value of k, for which the operator $\Delta_k + V$ must be injective. We set $k = \left\langle \frac{1}{2} + i\tau, 0, 0, \ldots, 0 \right\rangle$ with $\tau \in \mathbb{R}_+$. We get the injectivity of $\Delta_k + V$ by proving the inequality

$$\|u\|_{\mathscr{L}^2(\mathbb{T}^n)} \ll \frac{1}{\tau} \|(\Delta_k + V)\, u\|_{\mathscr{L}^2(\mathbb{T}^n)}, \qquad (\tau \longrightarrow \infty) \tag{$\mathscr{A}$}$$

which holds for all $u \in \mathscr{H}^2(\mathbb{T}^n)$.

We begin by considering the magnitude of the complex polynomial $(\xi + k)^2$. We have

$$\begin{aligned} \left|(\xi + k)^2\right| &= \left|(\xi + \Re k) \cdot (\xi + \Re k) - \Im k \cdot \Im k + 2i\,(\xi + \Re k) \cdot \Im k\right| \\ &\geqslant \left|2i\,(\xi + \Re k) \cdot \Im k\right| = \left|2i \left(\xi_1 + \frac{1}{2}\right) \tau\right| \geqslant \tau \end{aligned}$$

for all $\xi \in \mathbb{Z}^n$ and $\tau \in \mathbb{R}_+$. But since the operator Δ_k is just multiplication by $4\pi^2 (\xi + k)^2$ on the Fourier series side, we immediately get the resolvent inequality

$$\|u\|_{\mathscr{L}^2(\mathbb{T}^n)} \ll \frac{1}{\tau} \|\Delta_k u\|_{\mathscr{L}^2(\mathbb{T}^n)}, \tag{$\mathscr{B}$}$$

which holds for all functions $u \in \mathscr{H}^2(\mathbb{T}^n)$ and every $\tau \in \mathbb{R}_+$.

Now the estimate $(\mathscr{A})$ is obtained easily, for

$$\begin{aligned}\|u\|_{\mathscr{L}^2(\mathbb{T}^n)} &\ll \frac{1}{\tau}\|\Delta_k u\|_{\mathscr{L}^2(\mathbb{T}^n)} \leqslant \frac{1}{\tau}\|(\Delta_k + V)\,u\|_{\mathscr{L}^2(\mathbb{T}^n)} + \frac{1}{\tau}\|Vu\|_{\mathscr{L}^2(\mathbb{T}^n)} \\ &\leqslant \frac{1}{\tau}\|(\Delta_k + V)\,u\|_{\mathscr{L}^2(\mathbb{T}^n)} + \frac{\|V\|_{\mathscr{L}^\infty(\mathbb{T}^n)}}{\tau}\|u\|_{\mathscr{L}^2(\mathbb{T}^n)}\end{aligned}$$

for all functions $u \in \mathscr{H}^2(\mathbb{T}^n)$ and every $\tau \in \mathbb{R}_+$, and thus for sufficiently large τ, also

$$\|u\|_{\mathscr{L}^2(\mathbb{T}^n)} \ll \frac{1}{\tau}\|(\Delta_k + V)\,u\|_{\mathscr{L}^2(\mathbb{T}^n)}.$$

A more general version of Thomas' argument

The proof in the previous section was minimal in some sense. In this section we will achieve two tasks. First, we will prove the absence of eigenvalues also for periodic Schrödinger operators of the form $-\nabla \cdot G\nabla + V$ for any given positive-definite symmetric real $n \times n$-matrix G. As mentioned before, this is equivalent with the absence of eigenvalues also for Schrödinger operators of the form $-\Delta + V$, which are periodic with respect to some other lattice than $\mathbb{Z}^n$. Second, we will prove the analog of the estimate $(\mathscr{B})$ for substantially wider class of vectors k. Although not strictly needed in this chapter, this will prove to be useful in Sobolev's argument.

We begin by proving the suitable generalization of $(\mathscr{B})$.

Lemma. *Let G be some fixed positive-definite symmetric real $n \times n$-matrix. Set for any non-zero integer vector v and any positive real number τ the value of k to be*

$$k = \frac{v}{2\,|v|^2} + \frac{i\tau G^{-1}v}{|v|}.$$

Then

$$\|f\|_{\mathscr{L}^2(\mathbb{T}^n)} \ll \frac{|v|}{\tau}\|\nabla_k \cdot G\nabla_k f\|_{\mathscr{L}^2(\mathbb{T}^n)} \tag{$\mathscr{C}$}$$

for all functions $f \in \mathscr{H}^2(\mathbb{T}^n)$, all vectors $v \in \mathbb{Z}^n \setminus \{0\}$, and all positive real numbers τ.

This lemma is also just the resolvent estimate which follows immediately from the fact that on the Fourier series side, $\nabla_k \cdot G\nabla_k$ means just multiplication by the complex polynomial $4\pi^2\,(\xi + k) \cdot G\,(\xi + k)$, and the fact that

$$\begin{aligned}|(\xi + k) \cdot G\,(\xi + k)| &= |(\xi + \Re k) \cdot G\,(\xi + \Re k) - \Im k \cdot G\Im k + 2i\,(\xi + \Re k) \cdot G\Im k| \\ &\geqslant |2i\,(\xi + \Re k) \cdot G\Im k| = \left|2i\left(\xi + \frac{v}{2\,|v|^2}\right) \cdot \frac{\tau v}{|v|}\right| = \frac{\tau\,|2\xi \cdot v + 1|}{|v|} \geqslant \frac{\tau}{|v|},\end{aligned}$$

for all $\xi \in \mathbb{Z}^n$, $v \in \mathbb{Z}^n \setminus \{0\}$ and $\tau \in \mathbb{R}_+$.

Given a symmetric positive-definite $G \in \mathbb{R}^{n\times n}$, a fixed vector $v \in \mathbb{Z}^n \setminus \{0\}$ and a bounded electric potential V, we again deduce that

$$\begin{aligned}\|u\|_{\mathscr{L}^2(\mathbb{T}^n)} &\ll \frac{1}{\tau}\|\nabla_k \cdot G\nabla_k u\|_{\mathscr{L}^2(\mathbb{T}^n)} \\ &\leqslant \frac{1}{\tau}\|(\nabla_k \cdot G\nabla_k + V)\,u\|_{\mathscr{L}^2(\mathbb{T}^n)} + \frac{1}{\tau}\|Vu\|_{\mathscr{L}^2(\mathbb{T}^n)} \\ &\leqslant \frac{1}{\tau}\|(\nabla_k \cdot G\nabla_k + V)\,u\|_{\mathscr{L}^2(\mathbb{T}^n)} + \frac{\|V\|_{\mathscr{L}^\infty(\mathbb{R}^n)}}{\tau}\|u\|_{\mathscr{L}^2(\mathbb{T}^n)}\end{aligned}$$

for every $u \in \mathscr{H}^2(\mathbb{T}^n)$, and again we have for sufficiently large τ that

$$\|u\|_{\mathscr{L}^2(\mathbb{T}^n)} \ll \frac{1}{\tau}\|(\nabla_k \cdot G\nabla_k + V)\,u\|_{\mathscr{L}^2(\mathbb{T}^n)}.$$

We obtain the

Corollary. *Let $G \in \mathbb{R}^{n\times n}$ be symmetric and positive-definite, and let $V \in \mathscr{L}^\infty(\mathbb{R}^n)$ be periodic. Then the periodic Schrödinger operator $-\nabla \cdot G\nabla + V$, with domain $\mathscr{H}^2(\mathbb{R}^n)$, has no eigenvalues.*

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We point out that by using the above choice of k, one also obtains Sobolev's theorem for a constant metric G. The required changes are only cosmetic; one has to have sufficiently small constant c in the definition of the cut-off function ψ_1 (p. 59), and one possibly has to restrict to sufficiently large τ to have the estimates in the support of ψ_τ (p. 59).

It is somewhat more interesting to note that one can easily handle sufficiently smooth isotropic metrics, i.e. metrics of the form αG, where G is as before and α is a sufficiently smooth periodic scalar function taking only positive values. The reason is simple: for all $\varphi \in \mathscr{H}^2(\mathbb{R}^n)$, we have:

$$-\frac{1}{\sqrt{\alpha}}\nabla \cdot \left(\alpha\nabla\left(\frac{\varphi}{\sqrt{\alpha}}\right)\right) = -\Delta\varphi + \widetilde{V}\varphi,$$

where $\widetilde{V}$ is some scalar function. Since multiplication by $\frac{1}{\sqrt{\alpha}}$ is a bounded invertible operator with a bounded inverse, Thomas' argument goes through. It is also clear that Sobolev's proof goes through, provided that α is sufficiently smooth, say $\lceil n+s \rceil$ times continuously differentiable, where $s \in \left]-\frac{n}{2}, \infty\right[$ is the same as in the statement of Sobolev's theorem.

From now on, we concentrate exclusively on the case in which $\alpha \equiv 1$ and G is the identity matrix.

Small magnetic potentials

Here we briefly consider the problem of proving absence of eigenvalues for the magnetic Schrödinger operator $(-i\nabla + A)^2 + V$. The Floquet transform leads again to a family of operators, entire of type $\mathscr{A}$ with respect to each component of k. In what follows, we write the family in the form $\Delta_k + W \cdot \nabla_k + V$. As before, it again suffices to come up with a single value of $k \in \mathbb{C}^n$ for which the operator $\Delta_k + W \cdot \nabla_k + V$ must be injective.

The proofs of the previous sections used the fact that Δ_k is just multiplication by $4\pi^2 (\xi + k)^2$ and this expression is $\Omega(\tau)$ for well-chosen k dependent on the positive real parameter τ, and this gives us the factor $\frac{1}{\tau}$ which is enough to easily take care of any bounded electric potential V.

In analogous fashion one might think that $2\pi (\xi + k)$ would be of order $\sqrt{\tau}$, giving a factor $\frac{1}{\sqrt{\tau}}$ to play with, thereby managing arbitrary bounded magnetic potentials. However, the complex vector $2\pi (\xi + k)$ is also of the order $\Omega(\tau)$.

Thus, in order to fit Thomas' argument for magnetic potentials, we need some way of adjusting the magnetic potential. One natural idea would be to just conjugate the original operator $(-i\nabla + A)^2 + V$ with some suitably chosen non-vanishing scalar function, say $e^{i\varphi}$. A simple calculation gives

$$e^{-i\varphi} \left(-\Delta - 2iA \cdot \nabla - i\nabla \cdot A + A^2 + V\right) e^{i\varphi} = -\Delta + (-2iA + i\nabla\varphi) \cdot \nabla + \widetilde{V},$$

for some new electric potential $\widetilde{V}$.

Thus we can subtract arbitrary gradients from the magnetic potential in question. However, it is clear that this can not make the entire magnetic potential to vanish, or even make it arbitrarily small.

The rest of this chapter is dedicated to the proof of absence of eigenvalues for $(-i\nabla - A)^2 + V$ in the case that A is sufficiently small (in the sense that its components have $\mathscr{L}^\infty$-norms smaller than some given fixed constant). This observation is due to R. Hempel and I. Herbst [He&He1, He&He2].

The motivation for this discussion is educational, for the idea of Sobolev's argument is to conjugate the operator $(-i\nabla - A)^2 + V$, not by a scalar function, but by some invertible zeroeth order toroidal pseudodifferential operator, arbitrarily reducing the size of the coefficients of the first-order terms. Excluding the construction of the requisite pseudodifferential operators, Sobolev's argument is largely based on same ideas as the following proof.

——:——

To show how small magnetic potentials may be handled, we shall prove the following statement:

Proposition. *Let $W \in \mathscr{L}^\infty(\mathbb{T}^n; \mathbb{C}^n)$ and $V \in \mathscr{L}^\infty(\mathbb{T}^n)$. We write k for $\left\langle \frac{1}{2} + i\tau, 0, 0, \ldots, 0 \right\rangle$ for each $\tau \in \mathbb{R}_+$. Then, if W is smaller in the $\mathscr{L}^\infty$-norm than some constant, which only depends on n, then*

$$\left\|u\right\|_{\mathscr{L}^2(\mathbb{T}^n)} \ll \frac{1}{\tau} \left\| \left(\nabla_k^2 + W \cdot \nabla_k + V\right) u \right\|_{\mathscr{L}^2(\mathbb{T}^n)}$$

for every function $u \in \mathscr{H}^2(\mathbb{T}^n)$ and sufficiently large τ.

We begin by noticing that

$$\left\| \nabla_k \Delta_k^{-1} \right\|_{\mathscr{L}^2(\mathbb{T}^n) \longrightarrow \mathscr{L}^2(\mathbb{T}^n; \mathbb{C}^n)} \ll 1.$$

The proof of this claim is simple: Let $\ell \in \{1, 2, \ldots, n\}$. It is clear that the operator $(-i\partial_\ell + 2\pi k_\ell)\,\Delta_k^{-1}$ simply multiplies the ξ^{th} Fourier coefficient of its argument by

$$\frac{\xi_\ell + k_\ell}{2\pi \left(\xi + k\right)^2}.$$

Here $\Re \left(\xi_\ell + k_\ell\right) \ll |\xi|$. When $\tau^2 > \frac{1}{2}\xi^2$, we may estimate

$$\Re \left(\xi_\ell + k_\ell\right) \ll \tau \ll \left(\xi + k\right)^2.$$

On the other hand, when $\xi^2 \geqslant 2\tau^2$, we can estimate

$$\Re \left(\xi_\ell + k_\ell\right) \ll \sqrt{\xi^2} \ll \left(\xi + k\right)^2,$$

since $\Re \left(\xi + k\right)^2 = \left(\xi + k\right)^2 - \tau^2$, and $\xi^2 - \tau^2 \geqslant \frac{1}{2}\xi^2$. Finally, $\Im k_\ell \leqslant \tau \ll \left(\xi + k\right)^2$, and we get

$$\left\| W \cdot \nabla_k \Delta_k^{-1} \right\|_{\mathscr{L}^2(\mathbb{T}^n) \longrightarrow \mathscr{L}^2(\mathbb{T}^n)} \ll 1.$$

For the rest of the argument, we assume that $\left\| W \right\|_{\mathscr{L}^\infty(\mathbb{T}^n;\mathbb{C}^n)}$ is so small that the above operator norm is at most $\frac{1}{2}$.

Now, if we write $f = \left(\Delta_k + W \cdot \nabla_k + V\right) u$ and let $v = \Delta_k u$, then

$$\left\| u \right\|_{\mathscr{L}^2(\mathbb{T}^n)} \ll \frac{1}{\tau} \left\| v \right\|_{\mathscr{L}^2(\mathbb{T}^n)},$$

and

$$\begin{aligned} \left\| f \right\|_{\mathscr{L}^2(\mathbb{T}^n)} &= \left\| \left(\Delta_k + W \cdot \nabla_k + V\right) \Delta_k^{-1} v \right\|_{\mathscr{L}^2(\mathbb{T}^n)} \\ &= \left\| \left(1 + W \cdot \nabla_k \Delta_k^{-1} + V \Delta_k^{-1}\right) v \right\|_{\mathscr{L}^2(\mathbb{T}^n)}. \end{aligned}$$

But since

$$\left\| W \cdot \nabla_k \Delta_k^{-1} \right\|_{\mathscr{L}^2(\mathbb{T}^n) \longrightarrow \mathscr{L}^2(\mathbb{T}^n)} \leqslant \frac{1}{2}, \quad \text{and} \quad \left\| V \Delta_k^{-1} \right\|_{\mathscr{L}^2(\mathbb{T}^n) \longrightarrow \mathscr{L}^2(\mathbb{T}^n)} \ll \frac{1}{\tau},$$

the operator $\Upsilon = 1 + W \cdot \nabla_k \Delta_k^{-1} + V \Delta_k^{-1}$ must be invertible for large enough τ, and furthermore, we may take $\left\| \Upsilon^{\pm 1} \right\|_{\mathscr{L}^2(\mathbb{T}^n) \longrightarrow \mathscr{L}^2(\mathbb{T}^n)} \asymp 1$, by making τ sufficiently large. This gives us

$$\left\| u \right\|_{\mathscr{L}^2(\mathbb{T}^n)} \ll \frac{1}{\tau} \left\| v \right\|_{\mathscr{L}^2(\mathbb{T}^n)} \ll \frac{1}{\tau} \left\| \Upsilon v \right\|_{\mathscr{L}^2(\mathbb{T}^n)} = \frac{1}{\tau} \left\| f \right\|_{\mathscr{L}^2(\mathbb{T}^n)},$$

as required. Q.E.D.

Toroidal Pseudodifferential Operators Depending on a Parameter

In this chapter we will give a self-contained presentation of the elements of parameter-dependent toroidal pseudodifferential operators. Besides the basic notation and properties of symbols, the important results that will be referred to in the last chapter are the composition theorem and the $\mathscr{L}^2$-estimates.

We wish to warn the reader that, in order to keep the formulas clean, we use some non-standard notation. In particular, we shall overload the symbol ∂ to denote also finite differences, we shall be liberal regarding the parameter-dependence, and the notations $\langle\xi\rangle_\tau$ and $S^m_{\rho,\delta,\tau}$ are simply non-standard.

We also point out that, for future purposes, we develop the theory for more general symbols than we will actually need in the last chapter. However, this extra generality carries with it almost no extra cost, and in the other contexts where pseudodifferential conjugation is used to eliminate large first order terms, more general classes of symbols are actually needed.

For the most part, our discussion follows the presentation of parameter-independent toroidal pseudodifferential operators in the chapters 3 and 4 of the book [R&T] of M. Ruzhansky and V. Turunen. Our discussion of $\mathscr{L}^2$-estimates is based on the section II.§6 of M. E. Taylor's book [Tay]. A standard reference regarding parameter-dependent pseudodifferential operators in the Euclidean space is Shubin's book [Shu], which was also useful in the writing of this chapter.

Some discrete analysis

The calculus of finite differences

Let $\varphi\colon \mathbb{Z}^n \longrightarrow \mathbb{C}$ be some discrete function. We need a concept which controls the growth of such functions, but for obvious reasons we can not use derivatives. However, we can use the much simpler difference calculus. For each $\ell \in \{1, 2, \ldots, n\}$, we define the partial difference operator $\partial_\ell\colon \mathbb{C}^{\mathbb{Z}^n} \longrightarrow \mathbb{C}^{\mathbb{Z}^n}$ by the formula

$$\big(\partial_\ell\varphi\big)(\xi) \overset{\text{def}}{=} \varphi(\xi_1, \ldots, \xi_{\ell-1}, \xi_\ell{+}1, \xi_{\ell+1}, \ldots, \xi_n) - \varphi(\xi)\,,$$

which is to hold for all $\xi \in \mathbb{Z}^n$. We extend the notation of partial differences to multi-indices in the usual way: for any multi-index α, we set

$$\partial^\alpha \overset{\text{def}}{=} \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}.$$

It is easy to see that

$$\big(\partial^\alpha \varphi\big)(\xi) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} (-1)^{|\alpha-\beta|}\, \varphi(\xi+\beta)$$

for any $\xi \in \mathbb{Z}^n$.

The calculus of finite differences is in many ways analogous to ordinary differential and integral calculus — a fact which is emphasized in the presentation [R&T] of Ruzhansky and Turunen. Regarding differences and derivatives, there is, however, a minor difference. The classical derivative is a local operation, whereas the notion of finite differences is not. For instance, when taking differences, the support of a finitely supported non-constant function always expands.

The discrete Leibniz formula

For the sake of cleaner notation, we introduce for any $\ell \in \{1, 2, \ldots, n\}$ the **translation operator** $E_\ell\colon \mathbb{C}^{\mathbb{Z}^n} \longrightarrow \mathbb{C}^{\mathbb{Z}^n}$ via the formula

$$\big(E_\ell \varphi\big)(\xi) = \varphi(\xi_1, \ldots, \xi_{\ell-1}, \xi_\ell{+}1, \xi_{\ell+1}, \ldots, \xi_n)\,,$$

which is again to hold for all $\xi \in \mathbb{Z}^n$, for each function $\varphi\colon \mathbb{Z}^n \longrightarrow \mathbb{C}$. For all multi-indices α, we define, as might be expected,

$$E^\alpha = E_1^{\alpha_1} E_2^{\alpha_2} \cdots E_n^{\alpha_n}.$$

We note that

$$\big(E^\alpha \varphi\big)(\xi) = \varphi(\xi + \alpha)$$

for any $\xi \in \mathbb{Z}^n$. Using this notation, the formula for iterated finite differences takes the form

$$\partial^\alpha \varphi = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} (-1)^{|\alpha-\beta|}\, E^\beta \varphi.$$

The following is the discrete analog of the Leibniz formula.

The discrete Leibniz theorem. *Let $\varphi, \psi\colon \mathbb{Z}^n \longrightarrow \mathbb{C}$ be arbitrary functions, and let α be an arbitrary multi-index. Then*

$$\partial^\alpha(\varphi\psi) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \big(E^\beta \partial^{\alpha-\beta} \varphi\big)\, \partial^\beta \psi$$

This formula is again obvious, once we notice that

$$\begin{aligned}\big(\partial_\ell(\varphi\psi)\big)(\xi) &= \varphi(\xi_1, \ldots, \xi_\ell{+}1, \ldots, \xi_n)\, \psi(\xi_1, \ldots, \xi_\ell{+}1, \ldots, \xi_n) - \varphi(\xi)\, \psi(\xi) \\ &= \big(E_\ell \varphi\big)(\xi)\, (\partial_\ell \psi)(\xi) - \big(\partial_\ell \varphi\big)(\xi)\, \psi(\xi)\,.\end{aligned}$$

for any $\ell \in \{1, 2, \ldots, n\}$ and all $\xi \in \mathbb{Z}^n$.

The discrete Taylor theorem

Let $x \in \mathbb{C}$ and $n \in \mathbb{Z}_+ \cup \{0\}$. We shall write $x^{(n)}$ for the product

$$x\,(x-1)(x-2)\cdots(x-n+1)\,.$$

Similarly, for a multi-index α and vector $\vartheta \in \mathbb{C}^n$, we define

$$\vartheta^{(\alpha)} = \vartheta_1^{(\alpha_1)}\vartheta_2^{(\alpha_2)}\cdots\vartheta_n^{(\alpha_n)}.$$

We point out that

$$\frac{1}{\alpha!}\vartheta^{(\alpha)} = \binom{\vartheta}{\alpha}.$$

We also introduce another notation. Usually, we take sums such as $\sum_{\ell=0}^{-5}$ to mean zero. However, we introduce **for the rest of this section and for the rest of this section only** the convention that for negative integers n, the summation sign $\sum_{\ell=0}^{n-1}$ denotes the sum $-\sum_{\ell=-n}^{-1}$. Of course, $\sum_{\ell=0}^{-1}$ will just denote an empty sum which always vanishes.

Lemma. a) *Let $\vartheta \in \mathbb{Z}$ and let $\alpha \in \mathbb{Z}_+ \cup \{0\}$. Then*

$$\sum_{k_1=0}^{\vartheta-1}\sum_{k_2=0}^{k_1-1}\cdots\sum_{k_\alpha=0}^{k_{\alpha-1}-1} 1 = \binom{\vartheta}{\alpha}.$$

b) *Let $\vartheta \in \mathbb{Z}^n$ and let α be an arbitrary multi-index. Then*

$$\sum_{k_{1,1}=0}^{\vartheta_1-1}\sum_{k_{1,2}=0}^{k_{1,1}-1}\cdots\sum_{k_{1,\alpha_1}=0}^{k_{1,\alpha_1-1}-1}$$

$$\sum_{k_{2,1}=0}^{\vartheta_2-1}\sum_{k_{2,2}=0}^{k_{2,1}-1}\cdots\sum_{k_{2,\alpha_2}=0}^{k_{2,\alpha_2-1}-1}$$

$$\cdots\cdots\cdots$$

$$\sum_{k_{n,1}=0}^{\vartheta_n-1}\sum_{k_{n,2}=0}^{k_{n,1}-1}\cdots\sum_{k_{n,\alpha_n}=0}^{k_{n,\alpha_n-1}-1} 1 = \binom{\vartheta}{\alpha}.$$

For clarity, we denote the sum in part b) by

$$\sum_{k}^{\vartheta}{}^{(\alpha)}$$

in what follows.

It is obvious that the claim of part b) follows directly from that of part a). The part a), in turn, follows from an elementary consideration when ϑ is non-negative:

$$\sum_{k_1=0}^{\vartheta-1}\sum_{k_2=0}^{k_1-1}\cdots\sum_{k_\alpha=0}^{k_{\alpha-1}-1} 1 = \sum_{k_1=0}^{\vartheta-1}\sum_{k_2=0}^{k_1-1}\cdots\sum_{k_{\alpha-1}=0}^{k_{\alpha-2}-1}\binom{k_{\alpha-1}}{1}$$

$$= \sum_{k_1=0}^{\vartheta-1}\sum_{k_2=0}^{k_1-1}\cdots\sum_{k_{\alpha-2}=0}^{k_{\alpha-3}-1}\binom{k_{\alpha-2}}{2} = \ldots = \binom{\vartheta}{\alpha}.$$

All the intermediate steps are obvious from the Pascal triangle. The case of negative ϑ is analogous.

——:——

The following is the substantial result of this section. We emphasize that the important symbol calculus formulae are ultimately just its direct applications.

The discrete Taylor formula. *Let* $\varphi\colon \mathbb{Z}^n \longrightarrow \mathbb{C}$ *and* $N \in \mathbb{Z}_+$ *be arbitrary. Then the remainder* r_N*, defined by the formula*

$$r_N(\xi, \vartheta) = \varphi(\xi + \vartheta) - \sum_{|\alpha|<N} \frac{1}{\alpha!} \vartheta^{(\alpha)} \partial^\alpha \varphi(\xi)$$

for all $\xi, \vartheta \in \mathbb{Z}^n$*, satisfies the inequality*

$$\partial^\alpha r_N(\cdot, \vartheta) \ll_N \langle\vartheta\rangle^N \max_{\substack{|\beta|=N,\\ \nu\in C(\vartheta)}} \left|\partial^{\alpha+\beta} \varphi(\xi + \nu)\right|$$

for all $\xi, \vartheta \in \mathbb{Z}^n$ *and all multi-indices* α*. Here* $C(\vartheta)$ *denotes the cube*

$$\Big\{\nu \in \mathbb{Z}^n \;\Big|\; |\nu_1| \leqslant |\vartheta_1|, |\nu_2| \leqslant |\vartheta_2|, \ldots, |\nu_n| \leqslant |\vartheta_n|\Big\}.$$

Furthermore, the implicit constant may be taken to be independent of φ*.*

Before the proof, we make the remark that for fixed values of ξ and ϑ, the remainder term is just a finite sum of values of the translates of φ. This will be useful in the symbol calculus proofs since the function φ will then depend on x-variable as well and we need to know how differentiating φ with respect to x affects the error term.

Proof. Without loss of generality, we may assume that $\xi = 0$. This simplifies notation greatly. The main point is that the remainder r_N has the expression

$$r_N(\xi, \vartheta) = \sum_{|\alpha|=N} \sum_{k}^{\vartheta}{}^{(\alpha)} \big(\partial^\alpha \varphi\big)(\vartheta_1, \vartheta_2, \ldots, \vartheta_{m_\alpha-1}, k_{m_\alpha, \alpha_{m_\alpha}}, 0, 0, \ldots, 0)\,,$$

where we denote by m_α the smallest index $\ell \in \{1, 2, \ldots, n\}$ for which $\alpha_\ell \neq 0$. The claimed inequality follows from this inequality via the observation

$$\left|\partial^\alpha r_B(\cdot, \vartheta)\right| \leqslant \sum_{|\beta|=N} \frac{1}{\beta!} \vartheta^{(\beta)} \max_{\nu\in C(\vartheta)} \left|\big(\partial^{\alpha+\beta}\varphi\big)(\xi + \nu)\right|,$$

and the easily seen inequality $\vartheta^{(\beta)} \ll_N \langle\vartheta\rangle^N$.

The proof of the identity is by induction on N and it is challenging only notationally. In the case $N = 1$ the claim reduces to the identity

$$\begin{aligned}\varphi(\vartheta) - \varphi(0) = \sum_{\ell=1}^{n} \sum_{k_{\ell,1}=0}^{\vartheta_\ell - 1} \big(&\varphi(\vartheta_1, \ldots, \vartheta_{\ell-1}, k_{\ell,1}{+}1, 0, \ldots, 0) \\ &- \varphi(\vartheta_1, \ldots, \vartheta_{\ell-1}, k_{\ell,1}, 0, \ldots, 0)\big),\end{aligned}$$

which is clearly true.

Next, we make the induction assumption that the identity holds for some $N \in \mathbb{Z}_+$. In the following we will write δ_ℓ for $\langle 0, 0, \ldots, 0, 1, 0, 0, \ldots, 0\rangle$, where only the ℓ^{th} component is nonzero. We obtain the induction step by combining the induction assumption with the already proven case $N = 1$:

$$r_{N+1}(\xi, \vartheta) = r_N(\xi, \vartheta) - \sum_{|\alpha|=N} \frac{1}{\alpha!} \vartheta^{(\alpha)} \partial^\alpha \varphi(\xi)$$

$$= \sum_{|\alpha|=N} \sum_{k}^{\vartheta} {}^{(\alpha)} \Big(\big(\partial^\alpha \varphi\big)(\vartheta_1, \ldots, \vartheta_{m_\alpha - 1}, k_{m_\alpha, \alpha_{m_\alpha}}, 0, \ldots, 0) - \big(\partial^\alpha \varphi\big)(\xi) \Big)$$

$$= \sum_{|\alpha|=N} \sum_{k}^{\vartheta} {}^{(\alpha)} \sum_{\ell=1}^{m_\alpha} \sum_{k'=0}^{\vartheta_\ell - 1} \Big(\big(\partial_\alpha \varphi\big)(\vartheta_1, \ldots, \vartheta_{\ell-1}, k'+1, 0, \ldots, 0) - \big(\partial_\alpha \varphi\big)(\vartheta_1, \ldots, \vartheta_{\ell-1}, k', 0, \ldots, 0) \Big)$$

$$= \sum_{|\alpha|=N} \sum_{\ell=1}^{m_\alpha} \sum_{k}^{\vartheta} {}^{(\alpha+\delta_\ell)} \big(\partial^{\alpha+\delta_\ell} \varphi\big)(\vartheta_1, \ldots, \vartheta_{\ell-1}, k_{\ell, \alpha_\ell + 1}, 0, \ldots, 0)$$

$$= \sum_{|\alpha|=N+1} \sum_{k}^{\vartheta} {}^{(\alpha)} \big(\partial^\alpha \varphi\big)(\vartheta_1, \ldots, \vartheta_{m_\alpha - 1}, k_{m_\alpha, \alpha_{m_\alpha}}, 0, \ldots, 0)\,.$$

The parameter τ

Objects we shall be looking at in the rest of this text will generally depend on a parameter which we shall always denote by τ. The parameter τ takes values from a set $\Lambda \subseteq [1, \infty[$, which for all purposes can be considered fixed. Implicit variables taking values from Λ are also denoted by τ. The actual Λ we will use is described on page 58. For much of the time the τ-dependence will not be visible in our notation.

The parameter-dependent order function $\langle \xi \rangle_\tau$

When $\xi \in \mathbb{Z}^n$ and $\tau \in \Lambda$, we write $\langle \xi \rangle_\tau$ for $\sqrt{\tau^2 + \xi^2}$. The standard $\langle \xi \rangle$ is obtained as the special case $\tau = 1$.

Proposition. *Let $\eta \in \mathbb{R}^n$ be fixed. Then*

$$\langle \xi + \eta \rangle_\tau \ll_\eta \langle \xi \rangle_\tau$$

for $\xi \in \mathbb{Z}^n$ and $\tau \in \Lambda$.

The proof is elementary:

$$\tau^2 + (\xi + \eta)^2 \leqslant \tau^2 + 2\xi^2 + 2\eta^2 \leqslant \big(2 + 2\eta^2\big)\big(\tau^2 + \xi^2\big)\,.$$

We make two remarks. First, the inequality becomes uniform with respect to η when η is restricted to a compact set. Second, by symmetry, we get

$$\langle \xi + \eta \rangle_\tau \asymp_\eta \langle \xi \rangle_\tau\,.$$

This too can be taken to be uniform when η lives in a compact set.

The class of parameter-dependent Schwartz functions $\mathscr{S}_\tau$

We define the class $\mathscr{S}_\tau$ of (parameter-dependent, discrete) **Schwartz functions** as the class of functions $\varphi\colon \mathbb{Z}^n \times \Lambda \longrightarrow \mathbb{C}$ for which $\varphi \ll_m \langle\cdot\rangle_\tau^m$ for every $m \in \mathbb{R}$. For later use, we collect here two easy properties of $\mathscr{S}_\tau$. In later sections, we will use these without any warnings.

Proposition. *Let $\varphi\colon \mathbb{Z}^n \times \Lambda \longrightarrow \mathbb{C}$ be a uniformly bounded function for which $\varphi(\xi,\tau) = 0$ whenever $\xi \in \mathbb{Z}^n$ or $\tau \in \Lambda$ is large enough. Then $\varphi \in \mathscr{S}_\tau$.*

Let $M \in \mathbb{R}_+$ and $R \in \mathbb{R}_+$ be such that $|\varphi| \leqslant M$, and that $\varphi = 0$ when $\langle\xi\rangle_\tau \geqslant R$. Then for any $m \in \mathbb{R}_+$,

$$\left|\varphi\right| \leqslant MR^m \langle\xi\rangle_\tau^{-m}.$$

Proposition. *Let $\varphi \in \mathscr{S}_\tau$. Then also $\partial_\xi^\alpha \varphi \in \mathscr{S}_\tau$ for any multi-index α.*

This follows from the explicit formula for ∂_ξ^α and the fact that a translate of $\langle\xi\rangle_\tau$ is of the same order of magnitude as $\langle\xi\rangle_\tau$ itself:

$$\partial_\xi^\alpha \varphi = \sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} (-1)^{|\alpha-\beta|} E^\beta \varphi \ll_{\alpha m} \sum_{\beta\leqslant\alpha} \langle\xi\rangle_\tau^m \ll_\alpha \langle\xi\rangle_\tau^m.$$

$S^m_{\rho,\delta,\tau}$ and $\mathrm{Op}[S^m_{\rho,\delta,\tau}]$

In everything that follows, m will always denote a real number, and ρ and δ will always denote real numbers from the set $[0,1]$. The letter τ continues to denote a number from the set $\Lambda \subseteq [1,\infty[$.

Definitions and basic properties

For any $m \in \mathbb{R}$ and $\rho, \delta \in [0,1]$, the class $S^m_{\rho,\delta,\tau}$ consists of all functions

$$\sigma\colon \mathbb{T}^n \times \mathbb{Z}^n \times \Lambda \longrightarrow \mathbb{C}$$

for which the following two conditions hold:

* the restriction $\sigma(\cdot,\xi,\tau) : \mathbb{T}^n \longrightarrow \mathbb{C}$ is smooth for any fixed values of $\xi \in \mathbb{Z}^n$ and $\tau \in \Lambda$;
* for any multi-indices α and β, we must have

  $$\partial_\xi^\alpha \partial_x^\beta \sigma(x,\xi,\tau) \ll_{\alpha\beta} \langle\xi\rangle_\tau^{m-\rho|\alpha|+\delta|\beta|}$$

  for $x \in \mathbb{T}^n$, $\xi \in \mathbb{Z}^n$ and $\tau \in \Lambda$.

We call elements of $S^m_{\rho,\delta,\tau}$ (parameter-dependent) **toroidal symbols**. The sets $S^m_{\rho,\delta,\tau}$ are **symbol classes**.

Naturally we will be dealing with many symbols. In order to lighten the notation, we not only denote the parameter always by τ, but also let ξ denote the implicit $\mathbb{Z}^n$-variable and x denote the implicit $\mathbb{T}^n$-variable. For instance, for the above function σ and some multi-index α, the expression $\partial_x^\alpha \sigma$ means the ∂^α-derivate of σ with respect to first n variables (which live in $\mathbb{T}^n$), and the expression $\partial_\xi^\alpha \sigma$ would mean the ∂^α-differences of σ with respect to the next n variables (which live in $\mathbb{Z}^n$).

The (parameter-dependent, toroidal) **pseudodifferential operator** $\mathrm{Op}(\sigma)$ corresponding to the symbol $\sigma \in S^m_{\rho,\delta,\tau}$ is defined to be the parameter-dependent operator defined by the formula

$$\big(\mathrm{Op}(\sigma)\,\varphi\big)(x) = \sum_{\xi\in\mathbb{Z}^n} e(x\cdot\xi)\,\sigma(x,\xi)\,\widehat{\varphi}(\xi)$$

for all $x \in \mathbb{T}^n$ and every $\varphi \in \mathscr{C}^\infty(\mathbb{T}^n)$ for fixed values of τ. It is clear that the function $\mathrm{Op}(\sigma)\,\varphi$ belongs to $\mathscr{C}^\infty(\mathbb{T}^n)$. The set of all pseudodifferential operators corresponding to symbols in $S^m_{\rho,\delta,\tau}$ is denoted by $\mathrm{Op}\big[S^m_{\rho,\delta,\tau}\big]$.

Proposition.

a) *The class $S^m_{\rho,\delta,\tau}$ expands when m or δ is increased or ρ decreased.*

b) *If $\sigma \in S^m_{\rho,\delta,\tau}$ and α and β are arbitrary multi-indices, then*

$$\partial_\xi^\alpha \partial_x^\beta \sigma \in S^{m-\rho|\alpha|+\delta|\beta|}_{\rho,\delta,\tau} \quad \textit{and} \quad E_\xi^\alpha \sigma \in S^m_{\rho,\delta,\tau}.$$

c) *If $\sigma_1 \in S^{m_1}_{\rho,\delta,\tau}$ and $\sigma_2 \in S^{m_2}_{\rho,\delta,\tau}$ for some $m_1, m_2 \in \mathbb{R}$ and $\rho, \delta \in [0,1]$, then*

$$\sigma_1 + \sigma_2 \in S^{\max\{m_1,m_2\}}_{\rho,\delta,\tau} \quad \textit{and} \quad \sigma_1\sigma_2 \in S^{m_1+m_2}_{\rho,\delta,\tau}.$$

Part **a)** is true simply because the inequalities defining the symbol class in question weaken when m or δ are increased or ρ is decreased.

Claims of **b)** are again trivial, the second because the expression $\langle\xi\rangle_\tau$ is comparable to $E_\xi^\alpha \langle\xi\rangle_\tau$.

The first claim in **c)** is again trivial and the second follows from the discrete Leibniz rule since for any multi-indices α and β,

$$\begin{aligned}
\partial_\xi^\alpha \partial_x^\beta (\sigma_1\sigma_2) &= \sum_{\alpha'\leqslant\alpha} \binom{\alpha}{\alpha'} \sum_{\beta'\leqslant\beta} \binom{\beta}{\beta'} \partial_\xi^{\alpha'} \partial_x^{\beta'} \sigma_1 \cdot E^{\alpha'} \partial_\xi^{\alpha-\alpha'} \partial_x^{\beta-\beta'} \sigma_2 \\
&\ll_{\alpha\beta\sigma_1\sigma_2} \sum_{\alpha'\leqslant\alpha} \sum_{\beta'\leqslant\beta} \langle\xi\rangle_\tau^{m_1-\rho|\alpha'|+\delta|\beta'|} \langle\xi\rangle_\tau^{m_2-\rho|\alpha-\alpha'|+\delta|\beta-\beta'|} \\
&\ll_{\alpha\beta} \langle\xi\rangle_\xi^{m_1+m_2-\rho|\alpha|+\delta|\beta|}.
\end{aligned}$$

The infinitely smoothing operators $\mathrm{Op}\big[S_\tau^{-\infty}\big]$

We define the class $S_\tau^{-\infty} = S^{-\infty}(\mathbb{T}^n \times \mathbb{Z}^n; \Lambda)$ of **infinitely smoothing symbols** simply as the intersection

$$S_\tau^{-\infty} = \bigcap_{m\in\mathbb{R}} S^m_{\rho,\delta,\tau}.$$

This is clearly independent of ρ and δ. We shall see later that the corresponding operators $\mathrm{Op}\big[S_\tau^{-\infty}\big]$ are actually Fredholm integral operators with smooth kernels. The name infinitely smoothing comes from the fact that an infinitely smoothing operator extends to a bounded (in fact compact) linear operator between any two Sobolev spaces on the torus $\mathbb{T}^n$ (see section 4.3 in [R&T]).

We say that two toroidal symbols σ_1 and σ_2 are equal modulo $S_\tau^{-\infty}$ if their difference belong to the class $S_\tau^{-\infty}$.

The following result is useful since it allows to excise an arbitrary symbol near the origin leaving the behaviour for large $\langle\xi\rangle_\tau$ intact while leaving only an infinitely smoothing remainder.

Proposition. *Let $\varphi \in \mathscr{S}_\tau$ and $\sigma \in S^{m_0}_{\rho,\delta,\tau}$ for some $m_0 \in \mathbb{R}$. Then $\varphi\sigma \in S_\tau^{-\infty}$.*

Let $m \in \mathbb{R}$ be given. Then, by the discrete Leibniz formula

$$\begin{aligned}\partial_\xi^\alpha \partial_x^\beta(\varphi\sigma) &= \sum_{\gamma\leqslant\alpha} \binom{\alpha}{\gamma} \left(E^\gamma \partial_\xi^{\alpha-\gamma}\varphi\right)\left(\partial_\xi^\gamma \partial_x^\beta \sigma\right) \\ &\ll_{\alpha\beta m} \sum_{\gamma\leqslant\alpha} \langle\xi\rangle_\tau^{m-m_0+\rho|\gamma|-\delta|\beta|} \langle\xi\rangle_\tau^{m_0-\rho|\gamma|+\delta|\beta|} \ll_\alpha \langle\xi\rangle_\tau^{m}.\end{aligned}$$

Fourier coefficients of symbols

Let $\sigma \in S^m_{\rho,\delta,\tau}$. We shall have use for the Fourier coefficients of σ with respect to the x-variable. For any $\eta \in \mathbb{Z}^n$, we define

$$\widehat{\sigma}(\eta,\cdot) \stackrel{\text{def}}{=} \int_{\mathbb{T}^n} \sigma(x,\cdot)\, e(-x\cdot\eta)\,\mathrm{d}x.$$

As one might expect, the Fourier coefficients will have good properties not the least because σ is a smooth function.

Proposition. *Let $\sigma \in S^m_{\rho,\delta,\tau}$ and let $\ell \in \mathbb{Z}_+ \cup \{0\}$ be arbitrary. Then*

$$\partial_\xi^\alpha \widehat{\sigma}(\eta,\xi) \ll_{\alpha\ell} \langle\eta\rangle^{-\ell} \langle\xi\rangle_\tau^{m-\rho|\alpha|+\delta\ell}$$

for all $\xi,\eta \in \mathbb{Z}^n$ and for all multi-indices α.

The proof uses a standard integration by parts trick. First, let ℓ be even. Since

$$\left(1-\frac{1}{4\pi^2}\Delta\right)^{\frac{\ell}{2}} e(-x\cdot\eta) = \langle\eta\rangle^\ell e(-x\cdot\eta)$$

for $x \in \mathbb{T}^n$ and $\eta \in \mathbb{Z}^n$, and since all the terms of $\left(1-\frac{1}{4\pi^2}\Delta\right)^{\frac{\ell}{2}}$ are of even order, we have

$$\begin{aligned}\partial_\xi^\alpha \widehat{\sigma}(\eta,\xi) &= \int_{\mathbb{T}^n} \partial_\xi^\alpha \sigma(x,\xi)\, e(-x\cdot\eta)\,\mathrm{d}x \\ &= \langle\eta\rangle^{-\ell} \int_{\mathbb{T}^n} \left(\left(1-\tfrac{1}{4\pi^2}\Delta\right)^{\frac{\ell}{2}} \partial_\xi^\alpha\sigma\right)(x,\xi)\, e(-x\cdot\eta)\,\mathrm{d}x \\ &\ll_{\alpha\ell} \langle\eta\rangle^{-\ell} \langle\xi\rangle_\tau^{m-\rho|\alpha|+\delta\ell}.\end{aligned}$$

When ℓ is odd, the numbers $\ell \pm 1$ are even and therefore

$$\begin{aligned}\partial_\xi^\alpha \widehat{\sigma}(\eta,\xi) &\ll_{\alpha\ell} \sqrt{\langle\eta\rangle^{-\ell+1} \langle\xi\rangle_\tau^{m-\rho|\alpha|+\delta(\ell-1)} \langle\eta\rangle^{-\ell-1} \langle\xi\rangle_\tau^{m-\rho|\alpha|+\delta(\ell+1)}} \\ &= \langle\eta\rangle^{-\ell} \langle\xi\rangle_\tau^{m-\rho|\alpha|+\delta\ell}.\end{aligned}$$

Composing zeroeth order symbols with smooth functions

We shall have several occasions to use the following practical result on compositions of real-valued zeroeth order symbols with smooth functions. Our model is the lemma I.6.1 from Shubin's book [Shu], which states the analogous result for complex-valued Euclidean symbols. In our applications, the function f will be one of exp, sin, cos, and raising to the power $\pm\frac{1}{2}$.

Proposition. *Let $\sigma \in S^0_{\rho,\delta,\tau}$ be real-valued, and let $f \in \mathscr{C}^\infty(\mathbb{R})$. Then also $f \circ \sigma \in S^0_{\rho,\delta,\tau}$.*

Clearly $\big(f \circ \sigma\big)(\cdot, \xi, \tau)$ is smooth on $\mathbb{T}^n$ for any fixed values of $\xi \in \mathbb{Z}^n$ and $\tau \in \Lambda$. It is also clear that $f \circ \sigma \ll 1$, since the image of σ is contained in a compact set of $\mathbb{R}$ and as a continuous function f must be bounded on that compact set.

Let α be a non-zero multi-index. The powers σ^2, σ^3, … belong to $S^0_{\rho,\delta,\tau}$ and all differences of the form $E^\beta_\xi \sigma - \sigma$ (with $\beta \leqslant \alpha$, say) belong to $S^{-\rho}_{\rho,\delta,\tau}$. Using Taylor's formula we can now estimate

$$
\begin{aligned}
&\partial^\alpha_\xi \big(f \circ \sigma\big) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} (-1)^{|\alpha-\beta|} \Big(f \circ E^\beta_\xi \sigma\Big) \\
&= \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} (-1)^{|\alpha-\beta|} \left(\sum_{\ell=1}^{|\alpha|-1} \frac{\Big(E^\beta_\xi \sigma - \sigma\Big)^\ell}{\ell!} \Big(f^{(\ell)} \circ \sigma\Big) + \overbrace{\Big(E^\beta_\xi \sigma - \sigma\Big)^{|\alpha|}}^{\ll_\alpha \langle\xi\rangle_\tau^{-\rho|\alpha|}} \frac{f^{(|\alpha|)}(x_\beta)}{|\alpha|!} \right) \\
&= \sum_{\ell=1}^{|\alpha|-1} \frac{f^{(\ell)} \circ \sigma}{\ell!} \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} (-1)^{|\alpha-\beta|} \sum_{k=0}^{\ell} \binom{\ell}{k} \Big(E^\beta_\xi \sigma\Big)^k \sigma^{\ell-k} + O_\alpha\Big(\langle\xi\rangle_\tau^{-\rho|\alpha|}\Big) \\
&= \sum_{\ell=1}^{|\alpha|-1} \underbrace{\frac{f^{(\ell)} \circ \sigma}{\ell!}}_{\ll_\alpha 1} \sum_{k=0}^{\ell} \binom{\ell}{k} \underbrace{\sigma^{\ell-k} \cdot \partial^\alpha_\xi \sigma^k}_{\ll_\alpha 1 \cdot \langle\xi\rangle_\tau^{-\rho|\alpha|}} + O_\alpha\Big(\langle\xi\rangle_\tau^{-\rho|\alpha|}\Big) \ll_\alpha \langle\xi\rangle_\tau^{-\rho|\alpha|}.
\end{aligned}
$$

Here, of course, the symbols x_β denote some unknown real numbers from some compact set.

Let β be an arbitrary multi-index. Then, by the chain rule, the derivative $\partial^\beta_x \left(f \circ \sigma\right)$ is a finite sum of terms which, apart from constant factors, are of the form

$$
\Big(f^{(\ell)} \circ \sigma\Big) \cdot \partial^{\gamma_1}_x \sigma \cdot \partial^{\gamma_2}_x \sigma \cdot \ldots \cdot \partial^{\gamma_\ell}_x \sigma,
$$

where $\ell \in \mathbb{Z}_+$ is not larger than $|\beta|$, and $\gamma_1, \gamma_2, \ldots, \gamma_\ell$ are multi-indices such that $|\gamma_1| + |\gamma_2| + \ldots + |\gamma_\ell| = |\beta|$. Therefore each of the terms, and thus also the derivative itself, may be estimated from above by $O_\beta\Big(\langle\xi\rangle_\tau^{\delta|\beta|}\Big)$.

The required inequality

$$
\partial^\alpha_\xi \partial^\beta_x \left(f \circ \sigma\right) \ll_{\alpha\beta} \langle\xi\rangle_\tau^{-\rho|\alpha|+\delta|\beta|}
$$

is now obtained by combining the above ideas using the discrete Leibniz formula.

To estimate differences, it suffices to estimate derivatives

In the next chapter, we will need to show that certain specific expressions give rise to genuine parameter-dependent toroidal symbols. However, for the sake of clarity we shall estimate ξ-derivatives instead of ξ-differences.

The connection to the general theory is that there is an alternative way of defining pseudodifferential operators on the torus, in which the symbols are smooth functions on $\mathbb{T}^n \times \mathbb{R}^n$ instead of $\mathbb{T}^n \times \mathbb{Z}^n$, and in which the ∂_ξ^α in the definition of symbols is interpreted as differentiation. It turns out that as long as $\rho \neq 0$ the two different approaches give the same pseudodifferential operators. We only need the easy direction of this equivalence. For deeper discussion we refer to [R&T, ch. 4].

Lemma. *Let $\sigma\colon \mathbb{T}^n \times \mathbb{R}^n \times \Lambda \longrightarrow \mathbb{C}$ be such that the restricted function $\sigma(\cdot,\cdot,\tau)\colon \mathbb{T}^n \times \mathbb{R}^n \longrightarrow \mathbb{C}$ is smooth for every $\tau \in \Lambda$, and let $m \in \mathbb{R}$ and $\rho, \delta \in [0,1]$ be arbitrary. Suppose that σ satisfies the inequalities*

$$\partial_\xi^\alpha \partial_x^\beta \sigma(\xi, x, \tau) \ll_{\alpha\beta} \langle \xi \rangle_\tau^{m-\rho|\alpha|+\delta|\beta|}$$

on $\mathbb{T}^n \times \mathbb{R}^n \times \Lambda$ for all multi-indices α and β, where each ∂ means differentiation. Then the restriction $\sigma|_{\mathbb{T}^n \times \mathbb{Z}^n \times \Lambda}$ belongs to the class $S^m_{\rho,\delta,\tau}$.

The proof is an easy application of the mean value theorem and induction. For this proof only, we shall denote differentiation by ∂ and taking differences by $\widetilde{\partial}$.

By the assumptions we have made,

$$\partial_x^\beta \sigma \ll_\beta \langle \xi \rangle_\tau^{m+\delta|\beta|}$$

on $\mathbb{T}^n \times \mathbb{R}^n \times \Lambda$ for every multi-index β. We fix the value of β, with the intention to induct on $|\alpha|$ and prove that

$$\widetilde{\partial}_\xi^\alpha \partial_x^\beta \sigma \ll_{\alpha\beta} \langle \xi \rangle_\tau^{m-\rho|\alpha|+\delta|\beta|}$$

on $\mathbb{T}^n \times \mathbb{R}^n \times \Lambda$ for all multi-indices α and β. The claimed inequalities are obtained from this as special cases.

Let now α be a multi-index, let $\ell \in \{1, 2, \ldots, n\}$ and assume that we have already shown that

$$\widetilde{\partial}_\xi^\alpha \partial_x^\beta \sigma \ll \langle \xi \rangle_\tau^{m-\rho|\alpha|+\delta|\beta|}$$

on $\mathbb{T}^n \times \mathbb{R}^n \times \Lambda$.

Since $\partial_{\xi_\ell} \partial_x^\beta \sigma \ll \langle \xi \rangle_\tau^{m-\rho+\delta|\beta|}$, the induction assumption for the function $\widetilde{\partial}_\xi^\alpha \partial_x^\beta \sigma$ gives the estimate

$$\widetilde{\partial}_\xi^\alpha \partial_{\xi_\ell} \partial_x^\beta \sigma \ll \langle \xi \rangle_\tau^{m-\rho-\rho|\alpha|+\delta|\beta|} \,.$$

But now the mean value theorem says that on

$$\begin{aligned}\widetilde{\partial}_{\xi_\ell} \widetilde{\partial}_\xi^\alpha \partial_x^\beta \sigma &= \Big(\partial_{\xi_\ell} \widetilde{\partial}_\xi^\alpha \partial_x^\beta \sigma\Big)(\xi + \eta\delta_\ell, x, \tau) = \Big(\widetilde{\partial}_\xi^\alpha \partial_{\xi_\ell} \partial_x^\beta \sigma\Big)(\xi + \eta\delta_\ell, x, \tau) \\ &\ll \langle \xi + \eta\delta_\ell \rangle_\tau^{m-\rho-\rho|\alpha|+\delta|\beta|} \ll \langle \xi \rangle_\tau^{m-\rho-\rho|\alpha|+\delta|\beta|} \,,\end{aligned}$$

where $\delta_\ell = \langle 0, \ldots, 0, 1, 0, \ldots, 0 \rangle$, for some $\eta \in \,]0,1[$ which depends heavily on ξ.

Asymptotic expansions of symbols

Our next task is to prove the following fundamental result.

Theorem. *Let $\langle \sigma_\nu \rangle_{\nu=1}^\infty \subseteq \bigcup_{m\in\mathbb{R}} S^m_{\rho,\delta,\tau}$ be a sequence of symbols and let $\langle m_\nu \rangle_{\nu=1}^\infty$ be a strictly decreasing sequence of real numbers such that we have $\sigma_\nu \in S^{m_\nu}_{\rho,\delta,\tau}$ for each $\nu \in \mathbb{Z}_+$, and that $m_\nu \xrightarrow[\nu\to\infty]{} -\infty$. Then there exists a symbol $\sigma \in S^{m_1}_{\rho,\delta,\tau}$ such that*

$$\sigma - \sum_{\nu=1}^{\nu_0-1} \sigma_\nu \in S^{m_{\nu_0}}_{\rho,\delta,\tau}$$

for each $\nu_0 \in \mathbb{Z}_+$. Furthermore, this symbol is unique modulo $S^{-\infty}_\tau$.

Of course, we would like to set $\sigma = \sum_{\nu=1}^\infty \sigma_\nu$, but there is no guarantee for the meaningfulness of this series. The standard way around this obstacle is to excise each of the symbols σ_ν so that the series becomes well-defined. The parameter-dependent case differs from the parameter-independent case in that we must excise with respect to τ as well.

Let $\langle R_\nu \rangle_{\nu=1}^\infty$ be a sequence of positive reals such that $R_\nu \xrightarrow[\nu\to\infty]{} \infty$, and let $\chi \in \mathscr{C}^\infty(\mathbb{R}_+)$ be an excision function in the sense that $\chi\big|_{]0,\frac{1}{2}[} \equiv 0$, $\chi\big|_{[1,\infty[} \equiv 1$, and $\operatorname{Im}\chi = [0,1]$. We define

$$\sigma \overset{\text{def}}{=} \sum_{\nu=1}^\infty \chi\left(\frac{\langle \xi \rangle_\tau}{R_\nu}\right) \sigma_\nu.$$

This series is well-defined since for any values of τ and ξ the series only has finitely many non-zero terms. Furthermore, since the non-zero terms are smooth functions in x, so is σ. For clarity, we denote the excision multiplier by χ_ν in what follows.

It is useful to note that $1 - \chi_\nu \in S^{-\infty}_\tau$, since $1 - \chi_\nu$ vanishes when ξ or τ is large enough. Similarly, for non-zero multi-indices α, the difference $\partial^\alpha_\xi \chi_\nu$ also belong to $S^{-\infty}_\tau$, and consequently $\chi_\nu \in S^0_{\rho,\delta,\tau}$. This also guarantees that $\chi_\nu \sigma_\nu \in S^{m_\nu}_{\rho,\delta,\tau}$.

Our plan is to show that the sequence $\langle R_\nu \rangle_{\nu=1}^\infty$ may be chosen so that we have the inequality

$$\partial^\alpha_\xi \partial^\beta_x \left(\chi_\nu \sigma_\nu\right) \leqslant 2^{-\nu} \langle \xi \rangle_\tau^{m_\nu+1-\rho|\alpha|+\delta|\beta|} \tag{α}$$

for any $\nu \in \mathbb{Z}_+$ and any multi-indices α and β such that $\nu \geqslant |\alpha|$ and $\nu \geqslant |\beta|$. This requirement will lead to finitely many lower bounds for each number R_ν.

First we show how the inequalities (α) suffice to prove that σ has the required properties. Let $\nu_0 \in \mathbb{Z}_+$ be given. Then

$$\sigma - \sum_{\nu=1}^{\nu_0-1} \sigma_\nu = \sum_{\nu=1}^{\nu_0-1} \left(\chi_\nu - 1\right) \sigma_\nu + \sum_{\nu=\nu_0}^\infty \chi_\nu \sigma_\nu.$$

Since each of the functions $\chi_\nu - 1$ is vanishes for large enough ξ and τ, the first sum $\sum_{\nu=1}^{\nu_0-1} \ldots$ belongs to the class $S^{-\infty}_\tau$. Let α and β be given multi-indices.

We have to prove that

$$\partial_\xi^\alpha \partial_x^\beta \sum_{\nu=\nu_0}^{\infty} \chi_\nu \sigma_\nu \ll_{\alpha\beta} \langle \xi \rangle_\tau^{m_{\nu_0} - \rho|\alpha| + \delta|\beta|} .$$

Let $\nu_1 \in \mathbb{Z}_+$ be so large that $\nu_1 \geqslant |\alpha|$, $\nu_1 \geqslant |\beta|$ and $m_{\nu_1} + 1 \leqslant m_{\nu_0}$.

Since $\sum_{\nu=\nu_0}^{\nu_1 - 1} \chi_\nu \sigma_\nu \in S_{\rho,\delta,\tau}^{m_{\nu_0}}$, it suffices that we prove that

$$\partial_\xi^\alpha \partial_x^\beta \sum_{\nu=\nu_1}^{\infty} \chi_\nu \sigma_\nu \ll_{\alpha\beta} \langle \xi \rangle_\tau^{m_{\nu_0} - \rho|\alpha| + \delta|\beta|} .$$

But by our choice of ν_1 and the inequalities (α), we have

$$\left| \sum_{\nu=\nu_1}^{\infty} \partial_\xi^\alpha \partial_x^\beta (\chi_\nu \sigma_\nu) \right| \leqslant \sum_{\nu=\nu_1}^{\infty} 2^{-\nu} \langle \xi \rangle_\tau^{m_\nu + 1 - \rho|\alpha| + \delta|\beta|} \leqslant \langle \xi \rangle_\tau^{m_{\nu_0} - \rho|\alpha| + \delta|\beta|} .$$

We must next show that the inequalities (α) can really be made to hold. But we know already that for fixed $\nu \in \mathbb{Z}_+$, and any multi-indices α and β,

$$\partial_\xi^\alpha \partial_x^\beta (\chi_\nu \sigma_\nu) \ll_{\alpha\beta} \langle \xi \rangle_\tau^{m_\nu - \rho|\alpha| + \delta|\beta|} = \langle \xi \rangle_\tau^{-1} \langle \xi \rangle_\tau^{m_\nu + 1 - \rho|\alpha| + \delta|\beta|} . \qquad (\beta)$$

It therefore suffices that we show that by choosing R_ν to be large enough, the factor $\langle \xi \rangle_\tau^{-1}$, together with the implicit constant, may be approximated from above by $2^{-\nu}$. But by choosing R_ν to be much larger compared to $|\alpha|$, say $R_\nu \geqslant 3\,|\alpha|$, we know that χ_ν, all its differences $\partial_\xi^\gamma \chi_\nu$, and the translates of these differences that make their appearance in the discrete Leibniz formula for $\partial_\xi^\alpha \left(\chi_\nu \partial_x^\beta \sigma_\nu \right)$ vanish when $\langle \xi \rangle_\tau \leqslant \frac{1}{3} R_\nu$ or when $\langle \xi \rangle_\tau \geqslant 3R_\nu$. Thus, in the inequality (β), the factor $\langle \xi \rangle_\tau^{-1}$ may actually be taken proportional to R_ν, since the left-hand side expression vanishes otherwise. Now it is clear that by choosing R_ν large enough, we may compensate for the implicit constant and get (α). The existence of σ has now been proved.

Finally, the uniqueness modulo $S_\tau^{-\infty}$ is easy to see. If $\widetilde{\sigma} \in S_{\rho,\delta,\tau}^{m_1}$ is an another symbol for which

$$\widetilde{\sigma} - \sum_{\nu=1}^{\nu_0 - 1} \sigma_\nu \in S_{\rho,\delta,\tau}^{m_{\nu_0}}$$

for every $\nu_0 \in \mathbb{Z}_+$, then also

$$\sigma - \widetilde{\sigma} = \left(\sigma - \sum_{\nu=1}^{\nu_0 - 1} \sigma_\nu \right) - \left(\widetilde{\sigma} - \sum_{\nu=1}^{\nu_0 - 1} \sigma_\nu \right) \in S_{\rho,\delta,\tau}^{m_{\nu_0}}$$

for every $\nu_0 \in \mathbb{Z}_+$. Q.E.D.

Compositions

Next we shall prove an important theorem, which says that compositions of pseudodifferential operators are also pseudodifferential operators, their orders behave as one would expect, and their symbols have useful asymptotic expansions.

In this and the next section we will denote

$$D_\ell = \frac{1}{2\pi i}\frac{\partial}{\partial x_\ell}$$

for each $\ell \in \{1, 2, \ldots, n\}$,

$$D_\ell^{(k)} = D_\ell\,(D_\ell - 1)(D_\ell - 2)\cdots(D_\ell - k + 1)$$

for any $k \in \mathbb{Z}_+ \cup \{0\}$, and

$$D_x^{(\alpha)} = D_1^{(\alpha_1)} D_2^{(\alpha_2)} \cdots D_n^{(\alpha_n)}$$

for all multi-indices α.

Theorem. *Let $A \in \mathrm{Op}\big[S^{m_1}_{\rho,\delta,\tau}\big]$ and $B \in \mathrm{Op}\big[S^{m_2}_{\rho,\delta,\tau}\big]$ with $m_1, m_2 \in \mathbb{R}$ and $\rho > \delta$. Then the composed operator $A \circ B$ is a pseudodifferential operator in the class $\mathrm{Op}\big[S^{m_1+m_2}_{\rho,\delta,\tau}\big]$ and its symbol σ_{AB} has the asymptotic expansion*

$$\sigma_{AB} \sim \sum_{\alpha \geqslant 0} \frac{1}{\alpha!}\partial_\xi^\alpha \sigma_A \cdot D_x^{(\alpha)}\sigma_B,$$

where σ_A and σ_B are the symbols of A and B, respectively.

Let $\varphi \in \mathscr{C}^\infty(\mathbb{T}^n)$. Then for any $x \in \mathbb{T}^n$,

$$\begin{aligned}
\big(AB\varphi\big)(x) &= \sum_{\xi\in\mathbb{Z}^n} e(x\cdot\xi)\,\sigma_A(x,\xi)\,\widehat{B\varphi}(\xi) \\
&= \sum_{\xi\in\mathbb{Z}^n} e(x\cdot\xi)\,\sigma_A(x,\xi)\int_{\mathbb{T}^n}\sum_{\eta\in\mathbb{Z}^n} e(y\cdot\eta)\,\sigma_B(y,\eta)\,\widehat{\varphi}(\eta)\,e(-y\cdot\xi)\,\mathrm{d}y \\
&= \sum_{\eta\in\mathbb{Z}^n} e(x\cdot\eta)\sum_{\xi\in\mathbb{Z}^n} e\big(x\cdot(\xi-\eta)\big)\,\sigma_A(x,\xi) \\
&\qquad\qquad \cdot\int_{\mathbb{T}^n}\sigma_B(y,\eta)\,e\big(-y\cdot(\xi-\eta)\big)\,\mathrm{d}y\,\widehat{\varphi}(\eta) \\
&= \sum_{\eta\in\mathbb{Z}^n} e(x\cdot\eta)\sum_{\xi\in\mathbb{Z}^n} e\big(x\cdot(\xi-\eta)\big)\,\sigma_A(x,\xi)\,\widehat{\sigma_B}(\xi-\eta,\eta)\,\widehat{\varphi}(\eta) \\
&= \sum_{\eta\in\mathbb{Z}^n} e(x\cdot\eta)\sum_{\zeta\in\mathbb{Z}^n} e(x\cdot\zeta)\,\sigma_A(x,\zeta+\eta)\,\widehat{\sigma_B}(\zeta,\eta)\,\widehat{\varphi}(\eta)\,.
\end{aligned}$$

Thus AB is at least formally a pseudodifferential operator with symbol given by the formula

$$\sigma_{AB}(x,\xi) = \sum_{\eta\in\mathbb{Z}^n} e(x\cdot\eta)\,\sigma_A(x,\xi+\eta)\,\widehat{\sigma_B}(\eta,\xi)$$

for all $x \in \mathbb{T}^n$ and $\xi \in \mathbb{Z}^n$. Thus it suffices that we prove that $\sigma_{AB} \in S^{m_1+m_2}_{\rho,\delta,\tau}$ with the claimed asymptotic expansion.

Let $N \in \mathbb{Z}_+$. The discrete Taylor expansion allows us to write

$$\begin{aligned}\sigma_{AB}(x,\xi) &= \sum_{\eta\in\mathbb{Z}^n} e(x\cdot\eta)\,\sigma_A(x,\xi+\eta)\,\widehat{\sigma_B}(\eta,\xi)\\ &= \sum_{\eta\in\mathbb{Z}^n} e(x\cdot\eta) \sum_{|\alpha|<N}\frac{1}{\alpha!}\eta^{(\alpha)}\partial_\xi^\alpha\sigma_A(x,\xi)\,\widehat{\sigma_B}(\eta,\xi)\\ &\qquad\qquad + \sum_{\eta\in\mathbb{Z}^n} e(x\cdot\eta)\,r_N(x,\xi,\eta)\,\widehat{\sigma_B}(\eta,\xi)\\ &= \sum_{|\alpha|<N}\frac{1}{\alpha!}\partial_\xi^\alpha\sigma_A(x,\xi)\cdot D_x^{(\alpha)}\sigma_B(x,\xi)\\ &\qquad\qquad + \sum_{\eta\in\mathbb{Z}^n} e(x\cdot\eta)\,r_N(x,\xi,\eta)\,\widehat{\sigma_B}(\eta,\xi)\,.\end{aligned}$$

Here $r_N(x,\cdot,\cdot)$ is the error term given by the discrete Taylor formula for the function $\sigma_A(x,\cdot\ )$. That is, we have error terms corresponding to infinitely many applications of the discrete Taylor formula to infinitely many functions instead of merely infinitely many applications of the formula to a single function. This does not, however, cause any problems since the implicit constant in the estimate for the error term was immune to such things. (Cf. p. 37.)

Our goal is to prove that the error term can be made to belong to $S^{-M}_{0,0,\tau}$ for any $M \in \mathbb{Z}_+$ by taking N to be sufficiently large. Then we are done in view of the theorem on asymptotic expansions (cf. p. 44). So, let us be given $M \in \mathbb{Z}_+$. Given any multi-indices α and β, we may estimate using the properties of r_N and those of the Fourier coefficients of symbols (cf. p. 41) that

$$\begin{aligned}&\partial_\xi^\alpha\partial_x^\beta \sum_{\eta\in\mathbb{Z}^n} e(x\cdot\eta)\,r_N(x,\xi,\eta)\,\widehat{\sigma_B}(\eta,\xi)\\ &\quad= \sum_{\eta\in\mathbb{Z}^n}\sum_{\alpha'\leqslant\alpha}\binom{\alpha}{\alpha'}\sum_{\beta'\leqslant\beta}\binom{\beta}{\beta'}\partial_\xi^{\alpha'}\partial_x^{\beta'}\left(e(x\cdot\eta)\,\widehat{\sigma_B}(\eta,\xi)\right)\\ &\qquad\qquad\qquad\cdot\partial_\xi^{\alpha-\alpha'}\partial_x^{\beta-\beta'}r_N(x,\xi,\eta)\\ &\quad\ll_{\alpha\beta N\ell} \sum_{\eta\in\mathbb{Z}^n}\sum_{\alpha'\leqslant\alpha}\sum_{\beta'\leqslant\beta}\langle\eta\rangle^{|\beta'|}\langle\eta\rangle^{-\ell}\langle\xi\rangle_\tau^{m_2-\rho|\alpha'|+\delta\ell}\\ &\qquad\qquad\qquad\cdot\langle\eta\rangle^N \max_{\substack{|\gamma|=N,\\ \nu\in C(\eta)}}\left|\partial_\xi^{\alpha-\alpha'+\gamma}\partial_x^{\beta-\beta'}\sigma_A(x,\xi+\nu)\right|\\ &\quad\ll_{\alpha\beta N} \sum_{\eta\in\mathbb{Z}^n}\sum_{\alpha'\leqslant\alpha}\sum_{\beta'\leqslant\beta}\langle\eta\rangle^{|\beta'|+N-\ell}\langle\xi\rangle_\tau^{m_2-\rho|\alpha'|+\delta\ell}\\ &\qquad\qquad\qquad\cdot\max_{\nu\in C(\eta)}\langle\xi+\nu\rangle_\tau^{m_1-\rho N-\rho|\alpha-\alpha'|+\delta|\beta-\beta'|}\,.\end{aligned}$$

Here $\ell \in \mathbb{Z}_+$ may be chosen separately for each term.

For $|\eta| \leqslant \frac{\langle\xi\rangle_\tau}{2}$, we can estimate $\langle\xi+\nu\rangle_\tau \asymp \langle\xi\rangle_\tau$, and by choosing $\ell = N+|\beta'|$ in each term and letting N be large enough, we get

$$\begin{aligned}&\sum_{|\eta|\leqslant\frac{\langle\xi\rangle_\tau}{2}}\sum_{\alpha'\leqslant\alpha}\sum_{\beta'\leqslant\beta}\langle\eta\rangle^{|\beta'|+N-\ell}\langle\xi\rangle_\tau^{m_2-\rho|\alpha'|+\delta\ell}\max_{\nu\in C(\eta)}\langle\xi+\nu\rangle_\tau^{m_1-\rho N-\rho|\alpha-\alpha'|+\delta|\beta-\beta'|}\\ &\qquad\ll_{\alpha\beta N}\sum_{|\eta|\leqslant\frac{\langle\xi\rangle_\tau}{2}}\langle\xi\rangle_\tau^{m_1+m_2-(\rho-\delta)N-\rho|\alpha|+\delta|\beta|}\ll_{\alpha\beta M}\langle\xi\rangle_\tau^{-M}\,.\end{aligned}$$

On the other hand, when $|\eta| > \frac{\langle\xi\rangle_\tau}{2}$, we can choose $\ell = N+|\beta'|+M+(n+1)$

and estimate

$$\sum_{|\eta|>\frac{\langle\xi\rangle_\tau}{2}} \sum_{\alpha'\leqslant\alpha} \sum_{\beta'\leqslant\beta} \langle\eta\rangle^{|\beta'|+N-\ell} \langle\xi\rangle_\tau^{m_2-\rho|\alpha'|+\delta\ell} \max_{\nu\in C(\eta)} \langle\xi+\nu\rangle_\tau^{m_1-\rho N-\rho|\alpha-\alpha'|+\delta|\beta-\beta'|}$$

$$\begin{aligned} &\ll_{\alpha\beta M} \sum_{|\eta|>\frac{\langle\xi\rangle_\tau}{2}} \sum_{\alpha'\leqslant\alpha} \sum_{\beta'\leqslant\beta} \langle\eta\rangle^{-M-(n+1)} \\ &\qquad\qquad \cdot \max_{\nu\in C(\eta)} \langle\xi+\nu\rangle_\tau^{m_1+m_2-\rho|\alpha|-(\rho-\delta)N+\delta|\beta|+\delta M+\delta(n+1)} \\ &\ll_{\alpha\beta M} \langle\xi\rangle_\tau^{-M} \sum_{|\eta|>\frac{\langle\xi\rangle_\tau}{2}} \langle\eta\rangle^{-(n+1)} \cdot 1 \ll \langle\xi\rangle_\tau^{-M}, \end{aligned}$$

provided that N is large enough, and we are done.

Corollary. *Let $\sigma \in S^m_{\rho,\delta,\tau}$ with $\rho > \delta$. Then*

$$\frac{\partial}{\partial x_\ell} \circ \mathrm{Op}(\sigma) = \mathrm{Op}(\sigma) \circ \frac{\partial}{\partial x_\ell} + \mathrm{Op}\Big(\frac{\partial\sigma}{\partial x_\ell}\Big)$$

for any $\ell \in \{1, 2, \ldots, n\}$. In particular, $\frac{\partial}{\partial x_\ell} \circ \mathrm{Op}(\sigma) \in \mathrm{Op}\big[S^{m+1}_{\rho,\delta,\tau}\big]$.

Formal adjoints

Next we will consider formal adjoints of parameter-dependent toroidal pseudodifferential operators. After defining them, we prove that every operator $A \in \mathrm{Op}\big[S^m_{\rho,\delta,\tau}\big]$ with $\rho > \delta$ has a unique formal adjoint which is also an operator in $\mathrm{Op}\big[S^m_{\rho,\delta,\tau}\big]$ and that any principal symbol of the formal adjoint is the complex-conjugate of a principal symbol of A and vice versa. We will actually prove the full symbol calculus formula for asymptotic expansions of formal adjoints.

Let $A \in \mathrm{Op}\big[S^m_{\rho,\delta,\tau}\big]$ with $\rho > \delta$. Then a **formal adjoint** of A means a pseudodifferential operator A^* such that

$$\int_{\mathbb{T}^n} \overline{\varphi}\, A\psi = \int_{\mathbb{T}^n} \overline{A^*\varphi}\, \psi$$

for all functions $\varphi, \psi \in \mathscr{C}^\infty(\mathbb{T}^n)$. If A is a formal adjoint of itself, then we call A **formally self-adjoint**.

We remark immediately that since the set of smooth functions $\mathscr{C}^\infty(\mathbb{T}^n)$ is dense in $\mathscr{L}^2(\mathbb{T}^n)$, the operator A can have at most one formal adjoint. More precisely, if A^* and A' are two formal adjoints of A, then for any given functions $\varphi \in \mathscr{C}^\infty(\mathbb{T}^n)$ and $\psi \in \mathscr{C}^\infty(\mathbb{T}^n)$

$$\int_{\mathbb{T}^n} \overline{(A^* - A')\,\varphi}\, \psi = \int_{\mathbb{T}^n} \overline{\varphi}\, (A\psi - A\psi) = 0,$$

thereby proving that $(A^* - A')\,\varphi = 0$.

Another easy observation is that if A has a formal adjoint A^*, then A^* has A as its formal adjoint, simply because then we have for any $\varphi, \psi \in \mathscr{C}^\infty(\mathbb{T}^n)$ that

$$\int_{\mathbb{T}^n} \overline{\varphi}\, A^*\psi = \overline{\int_{\mathbb{T}^n} \overline{A^*\psi}\, \varphi} = \overline{\int_{\mathbb{T}^n} \overline{\psi}\, A\varphi} = \int_{\mathbb{T}^n} \overline{A\varphi}\, \psi.$$

Our main result in this section is the following standard symbol calculus formula.

Theorem. *Let $A \in \mathrm{Op}\big[S^m_{\rho,\delta,\tau}\big]$ with $\rho > \delta$. Then A has a unique formal adjoint A^* which is an operator in $\mathrm{Op}\big[S^m_{\rho,\delta,\tau}\big]$. Furthermore, the symbol σ_{A^*} of A^* has the asymptotic expansion*

$$\sigma_{A^*} \sim \sum_{\gamma \geqslant 0} \frac{1}{\gamma!} \partial_\xi^\gamma D_x^{(\gamma)} \overline{\sigma_A},$$

where σ_A denotes the symbol of A. In particular, if $\sigma \in S^m_{\rho,\delta,\tau}$ is a principal symbol of A, then $\overline{\sigma}$ is a principal symbol of A^*.

Let $\varphi, \psi \in \mathscr{C}^\infty(\mathbb{T}^n)$. Then

$$\begin{aligned}\int_{\mathbb{T}^n} \overline{\varphi(x)}\, \big(A\psi\big)(x)\, \mathrm{d}x &= \int_{\mathbb{T}^n} \overline{\varphi(x)} \sum_{\xi \in \mathbb{Z}^n} e(x\cdot\xi)\, \sigma_A(x,\xi) \int_{\mathbb{T}^n} \psi(y)\, e(-y\cdot\xi)\, \mathrm{d}y\, \mathrm{d}x \\ &= \int_{\mathbb{T}^n} \overline{\sum_{\xi\in\mathbb{Z}^n} \int_{\mathbb{T}^n} \varphi(x)\, \overline{\sigma_A(x,\xi)}\, e\big((y-x)\cdot\xi\big)\, \mathrm{d}x}\, \psi(y)\, \mathrm{d}y.\end{aligned}$$

Here all the exchanges of order of summation and integration are easily justified. We can continue to manipulate the expression under the large complex conjugation sign:

$$\begin{aligned}&\sum_{\xi\in\mathbb{Z}^n} \int_{\mathbb{T}^n} \varphi(x)\, \overline{\sigma_A(x,\xi)}\, e\big((y-x)\cdot\xi\big)\, \mathrm{d}x \\ &\quad= \sum_{\xi\in\mathbb{Z}^n} e(y\cdot\xi) \int_{\mathbb{T}^n} \varphi(x) \sum_{\eta\in\mathbb{Z}^n} \overline{\widehat{\sigma_A}(\eta,\xi)}\, e(-x\cdot\eta)\, e(-x\cdot\xi)\, \mathrm{d}x \\ &\quad= \sum_{\xi\in\mathbb{Z}^n} e(y\cdot\xi) \sum_{\eta\in\mathbb{Z}^n} \overline{\widehat{\sigma_A}(\eta,\xi)}\, \widehat{\varphi}(\xi+\eta) \\ &\quad= \sum_{\zeta\in\mathbb{Z}^n} e(y\cdot\zeta) \sum_{\eta\in\mathbb{Z}^n} e(-y\cdot\eta)\, \overline{\widehat{\sigma_A}(\eta,\zeta-\eta)}\, \widehat{\varphi}(\zeta) \\ &\quad= \sum_{\zeta\in\mathbb{Z}^n} e(y\cdot\zeta) \sum_{\eta\in\mathbb{Z}^n} e(-y\cdot\eta)\, \widehat{\overline{\sigma_A}}(-\eta,\zeta-\eta)\, \widehat{\varphi}(\zeta)\,.\end{aligned}$$

Thus, at least formally, A^* exists and has the symbol

$$\sigma_{A^*}(x,\xi) = \sum_{\eta\in\mathbb{Z}^n} e(x\cdot\eta)\, \widehat{\overline{\sigma_A}}(\eta,\xi+\eta)\,.$$

We only have to show that σ_{A^*} belongs to $S^m_{\rho,\delta,\tau}$ and, more precisely, has the claimed asymptotic expansion.

Our next step is to apply the discrete Taylor theorem to $\widehat{\overline{\sigma_A}}(\eta,\xi-\eta)$. Let $N \in \mathbb{Z}_+$. Then

$$\begin{aligned}\sigma_{A^*}(x,\xi) &= \sum_{\eta\in\mathbb{Z}^n} e(x\cdot\eta) \sum_{|\alpha|<N} \frac{1}{\alpha!} \eta^{(\alpha)} \partial_\xi^\alpha \widehat{\overline{\sigma_A}}(\eta,\xi) + \sum_{\eta\in\mathbb{Z}^n} e(x\cdot\eta)\, r_N(\xi,\eta) \\ &= \sum_{|\alpha|<N} \frac{1}{\alpha!} \partial_\xi^\alpha D_x^{(\alpha)} \overline{\sigma_A}(x,\xi) + \sum_{\eta\in\mathbb{Z}^n} e(x\cdot\eta)\, r_N(\xi,\eta)\,,\end{aligned}$$

where the factors $r_N(\xi,\eta)$ are the error terms given by the discrete Taylor theorem (cf. p. 37) for the functions $\widehat{\sigma_A}(\eta,\cdot)$. Thus, we have a family of error terms corresponding to infinitely many different functions, not the infinitely many error terms corresponding to one function. We will prove that the error term sum can be made to belong to any $S^{-M}_{0,0,\tau}$ for any given $M \in \mathbb{Z}_+$ simply by taking sufficiently large values for N. Then we are done in view of the theorem on asymptotic expansions of page 44.

Let α and β be arbitrary multi-indices. We will prove that

$$\partial_\xi^\alpha \partial_x^\beta \sum_{\eta\in\mathbb{Z}^n} e(x\cdot\eta)\, r_N(\xi,\eta) \ll_{\alpha\beta} \langle\xi\rangle_\tau^{-M},$$

provided that N is sufficiently large. The discrete Taylor theorem and the inequality concerning the Fourier coefficients of a symbol (cf. p. 41) give us the estimate

$$\begin{aligned}\partial_\xi^\alpha \partial_x^\beta \sum_{\eta\in\mathbb{Z}^n} e(x\cdot\eta)\, r_N(\xi,\eta) &\ll \sum_{\eta\in\mathbb{Z}^n} \langle\eta\rangle^{|\beta|} \left|\partial_\xi^\alpha r_N(\xi,\eta)\right| \\ &\ll_{\alpha N} \sum_{\eta\in\mathbb{Z}^n} \langle\eta\rangle^{|\beta|+N} \max_{\substack{|\gamma|=N,\\ \nu\in C(\eta)}} \left|\partial_\xi^{\alpha+\gamma}\widehat{\sigma_A}(\eta,\xi+\nu)\right| \\ &\ll_{\alpha N\ell} \sum_{\eta\in\mathbb{Z}^n} \langle\eta\rangle^{|\beta|+N-\ell} \max_{\nu\in C(\eta)} \langle\xi+\nu\rangle_\tau^{m-\rho N-\rho|\alpha|+\delta\ell}.\end{aligned}$$

Let us assume temporarily $|\eta| \leqslant \frac{\langle\xi\rangle_\tau}{2}$. The triangle inequality then guarantees that

$$\frac{1}{2}\langle\xi\rangle_\tau \leqslant \langle\xi+\eta\rangle_\tau \leqslant \frac{3}{2}\langle\xi\rangle_\tau,$$

and so, by choosing $\ell = N + |\beta|$ and letting N to be large enough,

$$\begin{aligned}&\sum_{|\eta|\leqslant\frac{\langle\xi\rangle_\tau}{2}} \langle\eta\rangle^{|\beta|+N-\ell} \max_{\nu\in C(\eta)} \langle\xi+\nu\rangle_\tau^{m-\rho N-\rho|\alpha|+\delta\ell} \\ &\qquad \ll_{\alpha\beta N} \sum_{|\eta|\leqslant\frac{\langle\xi\rangle_\tau}{2}} \langle\xi\rangle_\tau^{m-(\rho-\delta)N-\rho|\alpha|+\delta|\beta|} \ll_{\alpha\beta M} \langle\xi\rangle_\tau^{-M}.\end{aligned}$$

On the other hand, when $|\eta| > \frac{\langle\xi\rangle_\tau}{2}$, we choose $\ell = |\beta| + N - M - (n+1)$ and we estimate for large enough N that

$$\begin{aligned}&\sum_{|\eta|>\frac{\langle\xi\rangle_\tau}{2}} \langle\eta\rangle^{|\beta|+N-\ell} \max_{\nu\in C(\eta)} \langle\xi+\nu\rangle_\tau^{m-\rho N-\rho|\alpha|+\delta\ell} \\ &\quad = \sum_{|\eta|>\frac{\langle\xi\rangle_\tau}{2}} \langle\eta\rangle^{-M-(n+1)} \max_{\nu\in C(\eta)} \langle\xi+\nu\rangle_\tau^{m-(\rho-\delta)N-\rho|\alpha|+\delta|\beta|+\delta(M+(n+1))} \\ &\qquad \ll_{\alpha\beta M} \langle\xi\rangle_\tau^{-M} \sum_{|\eta|>\frac{\langle\xi\rangle_\tau}{2}} \langle\eta\rangle^{-(n+1)}\cdot 1 \ll \langle\xi\rangle_\tau^{-M}.\end{aligned}$$

——:——

As a corollary we obtain the following useful result which we shall use in the next section in proofs of $\mathscr{L}^2$-boundedness results.

Corollary. *Let $A \in \mathrm{Op}\big[S^m_{\rho,\delta,\tau}\big]$ $(\rho > \delta)$ be formally self-adjoint. Then A has a real-valued principal symbol. Furthermore, for any principal symbol σ of A, the real part $\Re\sigma$ is also a principal symbol of A.*

Let $\sigma \in S^m_{\rho,\delta,\tau}$ be a principal symbol of A. Then the symbol calculus formula for formal adjoints says that $A^* = A$ has also $\overline{\sigma}$ as another principal symbol. But then

$$A - \mathrm{Op}(\Re\sigma) = \frac{A - \mathrm{Op}(\sigma) + A - \mathrm{Op}(\overline{\sigma})}{2} \in \mathrm{Op}\big[S^{m-(\rho-\delta)}_{\rho,\delta,\tau}\big],$$

so that $\Re\sigma$ is a real-valued principal symbol for A.

$\mathscr{L}^2$-boundedness

In this final section on pseudodifferential operators, our goal is to prove that for $m \leqslant 0$ and $\rho > \delta$, any operator $A \in \mathrm{Op}\big[S^m_{\rho,\delta,\tau}\big]$ can be extended to a bounded linear operator of $\mathscr{L}^2(\mathbb{T}^n)$ with an operator norm bounded by $O(\tau^m)$.

$\mathscr{L}^2$-boundedness for operators in $\mathrm{Op}\big[S^{-\infty}_\tau\big]$

Our first goal is to prove that operators in $\mathrm{Op}\big[S^{-\infty}_\tau\big]$ extend to bounded operators of $\mathscr{L}^2(\mathbb{T}^n)$ with operator norm that may be estimated from above by $O_m(\tau^m)$ for any m. Since the symbol calculus developed in previous sections provides powerful tools which work modulo $S^{-\infty}_\tau$ it is easy to understand why this $\mathscr{L}^2$-estimate could be very useful.

We begin by proving that all operators in $\mathrm{Op}\big[S^{-\infty}_\tau\big]$ are in fact Fredholm integral operators with smooth kernels.

Proposition. *Let $\sigma \in S^{-\infty}_\tau$. Then there exists a function $K \in \mathscr{C}^\infty(\mathbb{T}^n \times \mathbb{T}^n)$, dependent on $\tau \in \Lambda$, such that*

$$\mathrm{Op}(\sigma)\,\varphi = \int_{\mathbb{T}^n} K(\cdot, y)\,\varphi(y)\,\mathrm{d}y$$

for every $\varphi \in \mathscr{C}^\infty(\mathbb{T}^n)$. Furthermore, this function K is given by the formula

$$K(x,y) = \sum_{\xi\in\mathbb{Z}^n} e\big((x-y)\cdot\xi\big)\,\sigma(x,\xi)$$

for any x and y in $\mathbb{T}^n$.

This is actually just a matter of direct computation. For any $x \in \mathbb{T}^n$:

$$\begin{aligned}\big(\mathrm{Op}(\sigma)\,\varphi\big)(x) &= \sum_{\xi\in\mathbb{Z}^n} e(x\cdot\xi)\,\sigma(x,\xi)\,\widehat{\varphi}(\xi)\\ &= \sum_{\xi\in\mathbb{Z}^n} e(x\cdot\xi)\,\sigma(x,\xi)\int_{\mathbb{T}^n}\varphi(y)\,e(-y\cdot\xi)\,\mathrm{d}y\\ &= \int_{\mathbb{T}^n}\sum_{\xi\in\mathbb{Z}^n} e\big((x-y)\cdot\xi\big)\,\sigma(x,\xi)\,\varphi(y)\,\mathrm{d}y.\end{aligned}$$

The inequalities defining $S^{-\infty}_\tau$ guarantee that everything converges absolutely and uniformly. In particular, the inner sum of the last line defines a smooth function of $\mathbb{T}^n \times \mathbb{T}^n$.

We can now tackle the $\mathscr{L}^2$-estimate.

Lemma. *Let $\sigma \in S_\tau^{-\infty}$. Then*

$$\big\|\mathrm{Op}(\sigma)\,\varphi\big\|_{\mathscr{L}^2(\mathbb{T}^n)} \ll_m \tau^m \big\|\varphi\big\|_{\mathscr{L}^2(\mathbb{T}^n)}$$

for any $\varphi \in \mathscr{C}^\infty(\mathbb{T}^n)$ and any nonpositive real number m.

We have by the inequalities of Minkowski and Cauchy–Schwarz–Bunyakovsky and the above proposition that

$$\big\|\mathrm{Op}(\sigma)\,\varphi\big\|_{\mathscr{L}^2(\mathbb{T}^n)} = \sqrt{\int_{\mathbb{T}^n}\bigg|\int_{\mathbb{T}^n} K(x,y)\,\varphi(y)\,\mathrm{d}y\bigg|^2 \mathrm{d}x} \leqslant \int_{\mathbb{T}^n}\sqrt{\int_{\mathbb{T}^n}\big|K(x,y)\,\varphi(y)\big|^2 \mathrm{d}x}\,\mathrm{d}y$$

$$= \int_{\mathbb{T}^n}\sqrt{\int_{\mathbb{T}^n}\big|K(x,y)\big|^2\mathrm{d}x}\,\big|\varphi(y)\big|\mathrm{d}y \leqslant \sqrt{\int_{\mathbb{T}^n}\int_{\mathbb{T}^n}\big|K(x,y)\big|^2\mathrm{d}x\,\mathrm{d}y}\sqrt{\int_{\mathbb{T}^n}\big|\varphi(y)\big|^2\mathrm{d}y}$$

for any $\varphi \in \mathscr{C}^\infty(\mathbb{T}^n)$. Thus it suffices that we prove the estimate

$$\big\|K\big\|_{\mathscr{L}^2(\mathbb{T}^n\times\mathbb{T}^n)} \ll_m \tau^m.$$

However, this follows immediately from the following estimate, which holds for all $x, y \in \mathbb{T}^n$ and with sufficiently large negative m:

$$|K(x,y)| \leqslant \sum_{\xi\in\mathbb{Z}^n}\big|\sigma(x,\xi)\big| \ll_m \sum_{\xi\in\mathbb{Z}^n}\langle\xi\rangle_\tau^m = \tau^m\sum_{\xi\in\mathbb{Z}^n}\left\langle\frac{\xi}{\tau}\right\rangle^m \leqslant \tau^m\sum_{\xi\in\mathbb{Z}^n}\langle\xi\rangle^m \ll_m \tau^m.$$

$\mathscr{L}^2$-boundedness for operators in $\mathrm{Op}\big[S^0_{\rho,\delta,\tau}\big]$

We begin with a lemma from which the case $m = 0$ easily follows. In the following proof we will be using rather freely the symbol calculus developed in previous sections.

Lemma. *Let $C \in \mathrm{Op}\big[S^0_{\rho,\delta,\tau}\big]$ be formally self-adjoint, $\rho > \delta$, and suppose C has a principal symbol $\sigma \in S^0_{\rho,\delta,\tau}$ which only takes positive real values and which is bounded away from zero. Then there exists a pseudodifferential operator $B \in \mathrm{Op}\big[S^0_{\rho,\delta,\tau}\big]$ such that*

$$C - B^*B \in \mathrm{Op}\big[S_\tau^{-\infty}\big].$$

The idea of the proof is to recursively construct an asymptotic expansion $\langle b_\nu\rangle_{\nu=1}^\infty$ for the symbol of B. Here we will have $b_\nu \in S^{-(\nu-1)(\rho-\delta)}_{\rho,\delta,\tau}$ for every $\nu \in \mathbb{Z}_+$.

Our starting point is to set

$$b_1 = \sqrt{\sigma} \in S^0_{\rho,\delta,\tau}.$$

This is correct since the the symbol σ is bounded away from zero and hence the square root function is smooth in a neighbourhood of the image of σ. We also define

$$R_1 = C - \mathrm{Op}(b_1)^*\,\mathrm{Op}(b_1) \in \mathrm{Op}\big[S^{-(\rho-\delta)}_{\rho,\delta,\tau}\big].$$

Note that R_1 is formally self-adjoint.

We make the induction assumption that we have for some $\nu_0 \in \mathbb{Z}_+$ symbols

$$b_1 \in S^0_{\rho,\delta,\tau}, \quad b_2 \in S^{-(\rho-\delta)}_{\rho,\delta,\tau}, \quad \ldots, \quad b_{\nu_0} \in S^{-(\nu_0-1)(\rho-\delta)}_{\rho,\delta,\tau},$$

and remainder operators

$$R_1 \in \operatorname{Op}\big[S^{-(\rho-\delta)}_{\rho,\delta,\tau}\big], \quad R_2 \in \operatorname{Op}\big[S^{-2(\rho-\delta)}_{\rho,\delta,\tau}\big], \quad \ldots, \quad R_{\nu_0} \in \operatorname{Op}\big[S^{-\nu_0(\rho-\delta)}_{\rho,\delta,\tau}\big],$$

such that

$$C - \operatorname{Op}(b_1 + b_2 + \ldots + b_\nu)^* \operatorname{Op}(b_1 + b_2 + \ldots + b_\nu) = R_\nu$$

for every $\nu \in \{1, 2, \ldots, \nu_0\}$.

Our task is to construct a symbol $b_{\nu_0+1} \in S^{-\nu_0(\rho-\delta)}_{\rho,\delta,\tau}$ such that the remainder

$$R_{\nu_0+1} = C - \operatorname{Op}(b_1 + b_2 + \ldots + b_{\nu_0} + b_{\nu_0+1})^* \operatorname{Op}(b_1 + b_2 + \ldots + b_{\nu_0} + b_{\nu_0+1})$$

belongs to the class $\operatorname{Op}\big[S^{-(\nu_0+1)(\rho-\delta)}_{\rho,\delta,\tau}\big]$. This is easily achieved by setting

$$b_{\nu_0+1} = \frac{r_{\nu_0}}{2b_1} = \frac{r_{\nu_0}}{2\sqrt{\sigma}},$$

where r_{ν_0} denotes a real-valued principal symbol of R_{ν_0}. Such a real-valued symbol must exist since the operator R_{ν_0} is formally self-adjoint. Now we have $b_{\nu_0+1} \in S^{-\nu_0(\rho-\delta)}_{\rho,\delta,\tau}$ and modulo $\operatorname{Op}\big[S^{-(\nu_0+1)(\rho-\delta)}_{\rho,\delta,\tau}\big]$ it so happens that

$$\begin{aligned}
C &- \operatorname{Op}(b_1 + \ldots + b_{\nu_0} + b_{\nu_0+1})^* \operatorname{Op}(b_1 + \ldots + b_{\nu_0} + b_{\nu_0+1}) \\
&\equiv C - \operatorname{Op}(b_1 + \ldots + b_{\nu_0})^* \operatorname{Op}(b_1 + \ldots + b_{\nu_0}) \\
&\qquad - \operatorname{Op}(b_{\nu_0+1})^* \operatorname{Op}(b_1) - \operatorname{Op}(b_1)^* \operatorname{Op}(b_{\nu_0+1}) \\
&= R_{\nu_0} - \operatorname{Op}\big(\overline{b_{\nu_0+1}} b_1 + b_1 b_{\nu_0+1}\big) \equiv 0.
\end{aligned}$$

We have finished our recursive construction.

Finally the formulae in the induction assumptions which now hold for every $\nu \in \mathbb{Z}_+$ amount to the fact that the pseudodifferential operator B constructed from the symbols $\langle b_\nu \rangle_{\nu=1}^\infty$ satisfies

$$C - B^* B \in \operatorname{Op}\big[S^{-\nu(\rho-\delta)}_{\rho,\delta,\tau}\big]$$

for every $\nu \in \mathbb{Z}_+$, and thus

$$C - B^* B \in \operatorname{Op}\big[S^{-\infty}_\tau\big].$$

Theorem. *Let $\sigma \in S^0_{\rho,\delta,\tau}$ with $\rho > \delta$. Then the pseudodifferential operator* $\operatorname{Op}(\sigma)$ *satisfies the inequality*

$$\big\|\operatorname{Op}(\sigma)\,\varphi\big\|_{\mathscr{L}^2(\mathbb{T}^n)} \ll \big\|\varphi\big\|_{\mathscr{L}^2(\mathbb{T}^n)}$$

for all functions $\varphi \in \mathscr{C}^\infty(\mathbb{T}^n)$. In other words, the operator $\operatorname{Op}(\sigma)$ *from $\mathscr{C}^\infty(\mathbb{T}^n)$ to itself extends to a bounded operator from $\mathscr{L}^2(\mathbb{T}^n)$ to itself for any fixed value of τ, and the operator norm is uniformly bounded with respect to τ.*

Let $M \in \mathbb{R}_+$ be such that $|\sigma(x,\xi,\tau)| \leqslant M$ for all $x \in \mathbb{T}^n$, $\xi \in \mathbb{Z}^n$ and $\tau \in \Lambda$. Also, let ε be some positive real number. Since the operator

$$\varepsilon + M^2 - \operatorname{Op}(\sigma)^* \operatorname{Op}(\sigma) \in \operatorname{Op}\big[S^0_{\rho,\delta,\tau}\big]$$

is formally self-adjoint and has principal symbol $\varepsilon + M^2 - |\sigma|^2$, which is real-valued and bounded from below by ε, the previous lemma says that there exist operators $B \in \operatorname{Op}\big[S^0_{\rho,\delta,\tau}\big]$ and $R \in \operatorname{Op}\big[S^{-\infty}_\tau\big]$ such that

$$\varepsilon + M^2 - \operatorname{Op}(\sigma)^* \operatorname{Op}(\sigma) = B^*B + R.$$

But R is an infinitely smoothing operator and thus uniformly bounded so that

$$\begin{aligned}\big\|\operatorname{Op}(\sigma)\,\varphi\big\|^2_{\mathscr{L}^2(\mathbb{T}^n)} &\leqslant \big\|\operatorname{Op}(\sigma)\,\varphi\big\|^2_{\mathscr{L}^2(\mathbb{T}^n)} + \big\|B\varphi\big\|^2_{\mathscr{L}^2(T^n)} \\ &= \big\langle\varphi\big|\operatorname{Op}(\sigma)^*\operatorname{Op}(\sigma)\,\varphi\big\rangle + \langle\varphi|B^*B\varphi\rangle \\ &= \big(\varepsilon + M^2\big)\,\langle\varphi|\varphi\rangle - \langle\varphi|R\varphi\rangle \ll \big\|\varphi\big\|^2_{\mathscr{L}^2(\mathbb{T}^n)}\end{aligned}$$

for any $\varphi \in \mathscr{C}^\infty(\mathbb{T}^n)$.

$\mathscr{L}^2$-boundedness for $\operatorname{Op}\big(\langle\xi\rangle^m_\tau\big)$

Proposition. *Let $m \in \mathbb{R}$ and let $\rho,\delta \in [0,1]$ be arbitrary. Then the function $\langle\xi\rangle^m_\tau$ belongs to the symbol class $S^m_{\rho,\delta,\tau}$.*

More precisely, we show that $\langle\xi\rangle^m_\tau \in S^m_{1,0,\tau}$. The function $\langle\xi\rangle^m_\tau$ is clearly a uniformly bounded smooth function for any fixed values of $\xi \in \mathbb{Z}^n$ and $\tau \in \Lambda$. Furthermore, the x-derivatives clearly vanish, and so it only remains to show that

$$\partial^\alpha_\xi \langle\xi\rangle^m_\tau \ll_\alpha \langle\xi\rangle^{m-|\alpha|}_\tau$$

for arbitrary multi-index α. But this is straightforward since $\langle\xi\rangle_\tau$ has the same asymptotic behaviour as its translates:

$$\begin{aligned}\partial^\alpha_\xi \langle\xi\rangle^m_\tau &= \sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} (-1)^{|\alpha-\beta|}\,\langle\xi+\beta\rangle^m_\tau \\ &= \langle\xi\rangle^{m-|\alpha|}_\tau \sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} (-1)^{|\alpha-\beta|} \left(\frac{\langle\xi+\beta\rangle^m_\tau}{\langle\xi\rangle^{m-|\alpha|}_\tau} - \langle\xi\rangle^{|\alpha|}_\tau\right) \\ &\ll_\alpha \langle\xi\rangle^{m-|\alpha|}_\tau \sum_{\beta\leqslant\alpha} 1 \ll_\alpha \langle\xi\rangle^{m-|\alpha|}_\tau .\end{aligned}$$

Proposition. *Let $m \in]-\infty,0]$. Then the operator $\operatorname{Op}\big(\langle\xi\rangle^m_\tau\big)$ satisfies the norm estimate*

$$\big\|\operatorname{Op}\big(\langle\xi\rangle^m_\tau\big)\,\varphi\big\|_{\mathscr{L}^2(\mathbb{T}^n)} \ll \tau^m \big\|\varphi\big\|_{\mathscr{L}^2(\mathbb{T}^n)}$$

for all $\varphi \in \mathscr{C}^\infty(\mathbb{T}^n)$.

This is easy: for any $\varphi \in \mathscr{C}^\infty(\mathbb{T}^n)$,

$$\left\|\mathrm{Op}\big(\langle\xi\rangle_\tau^m\big)\,\varphi\right\|^2_{\mathscr{L}^2(\mathbb{T}^n)} = \sum_{\xi\in\mathbb{Z}^n} \langle\xi\rangle_\tau^{2m}\left|\widehat{\varphi}(\xi)\right|^2 \leqslant \tau^{2m}\sum_{\xi\in\mathbb{Z}^n}\left|\widehat{\varphi}(\xi)\right|^2 = \tau^{2m}\left\|\varphi\right\|^2_{\mathscr{L}^2(T^n)}.$$

$\mathscr{L}^2$-boundedness for $\mathrm{Op}\big[S^m_{\rho,\delta,\tau}\big]$ with nonpositive m

We finish the chapter with the following

Theorem. *Let $\sigma \in S^m_{\rho,\delta,\tau}$ with $m \leqslant 0$ and $\rho > \delta$. Then*

$$\left\|\mathrm{Op}(\sigma)\,\varphi\right\|_{\mathscr{L}^2(\mathbb{T}^n)} \ll \tau^m\left\|\varphi\right\|_{\mathscr{L}^2(\mathbb{T}^n)}$$

for any $\varphi \in \mathscr{C}^\infty(\mathbb{T}^n)$.

This follows easily from the $\mathscr{L}^2$-boundedness properties of $\mathrm{Op}\big[S^0_{\rho,\delta,\tau}\big]$ and our knowledge of the operator $\mathrm{Op}\big(\langle\xi\rangle_\tau^m\big)$. Namely, we have

$$\mathrm{Op}(\sigma) = \mathrm{Op}\big(\langle\xi\rangle_\tau^m\big)\,\mathrm{Op}\big(\langle\xi\rangle_\tau^{-m}\big)\,\mathrm{Op}(\sigma)\,,$$

and the composition formula for pseudodifferential operators tells us that the composition $\mathrm{Op}\big(\langle\xi\rangle_\tau^{-m}\big)\circ\mathrm{Op}(\sigma)$ is actually an operator in the class $\mathrm{Op}\big[S^0_{\rho,\delta,\tau}\big]$ and therefore extends to a uniformly bounded operator of $\mathscr{L}^2(\mathbb{T}^n)$. On the other hand, we know that $\mathrm{Op}\big(\langle\xi\rangle_\tau^m\big)$ extends to a bounded operator of $\mathscr{L}^2(\mathbb{T}^n)$ with operator norm bounded by $O(\tau^m)$. Combining these two results gives the theorem.

Sobolev's Argument

In this final chapter we prove the absence of eigenvalues for the periodic magnetic Schrödinger operator with a bounded electric potential and a magnetic potential which is a sum of a vector-valued function whose components are trigonometric polynomials and a bounded vector-valued potential whose $\mathscr{L}^\infty$-norm is smaller than a small constant which depends on the trigonometric polynomials.

We begin with an informal presentation of the high-level structure of the proof, and the following three sections cover the necessary details: first the choice of parameters and some basic inequalities, then the definition of the intertwining operators. The final section combines all the lemmas to a final estimate which proves the absence of eigenvalues.

The proof is originally due to A. V. Sobolev [So1], who proved the result for sufficient smooth general magnetic potentials. It was subsequently used in the papers [Ku&L1, Ku&L2] to handle much more general classes of operators. Our presentation is based on M. Salo's unpublished note [Sa2].

An overview of the proof

Our goal is to prove the following special case of Sobolev's theorem:

Theorem. *Let $V \in \mathscr{L}^\infty(\mathbb{R}^n)$ be periodic and let $A^\sharp \in \mathscr{C}^\infty(\mathbb{R}^n;\mathbb{C}^n)$, suppose that the components of $A^\sharp$ are trigonometric polynomials, and suppose that $\widehat{A^\sharp}(0) = 0$. Furthermore, let $A^\flat \in \mathscr{L}^\infty(\mathbb{R}^n;\mathbb{C}^n)$ be periodic and satisfy*

$$\|A^\flat\|_{\mathscr{L}^\infty(\mathbb{R}^n;\mathbb{C}^n)} \leqslant \mathscr{J}_{A^\sharp},$$

where $\mathscr{J}_{A^\sharp} \in \mathbb{R}_+$ is a veritably small number which only depends on $A^\sharp$. Then the periodic magnetic Schrödinger operator

$$\Big(-i\nabla - A^\sharp - A^\flat\Big)^2 + V \colon \mathscr{H}^2(\mathbb{R}^n) \longrightarrow \mathscr{L}^2(\mathbb{R}^n)$$

has no eigenvalues.

Thanks to the approach of Thomas, it suffices that we prove the following norm estimate for well-chosen fibers of the Floquet transform of the operator:

Theorem. *Let $W^\sharp \in \mathscr{C}^\infty(\mathbb{T}^n;\mathbb{C}^n)$ have trigonometric polynomials as its components, and suppose that $\widehat{W^\sharp}(0) = 0$. Furthermore, let $V \in \mathscr{L}^\infty(\mathbb{T}^n)$ and let $W^\flat \in \mathscr{L}^\infty(\mathbb{T}^n;\mathbb{C}^n)$ satisfy*

$$\|W^\flat\|_{\mathscr{L}^\infty(\mathbb{T}^n;\mathbb{C}^n)} \leqslant \mathscr{J}_{W^\sharp},$$

where $\mathscr{J}_{W^\sharp} \in \mathbb{R}_+$ is a veritably small number which only depends on $W^\sharp$. Then there exists an unbounded subset $\Lambda \subset [1, \infty[$ and a function $k = k(\tau)$ from Λ to $\mathbb{C}^n$ such that

$$\left\|u\right\|_{\mathscr{L}^2(\mathbb{T}^n)} \ll_{W^\sharp} \frac{1}{\tau}\left\|\left(\Delta_k + W \cdot \nabla_k + V\right) u\right\|_{\mathscr{L}^2(\mathbb{T}^n)} \qquad (\mathscr{E})$$

for all functions $u \in \mathscr{H}^2(\mathbb{T}^n)$ and sufficiently large values of $\tau \in \Lambda$, where $W = W^\sharp + W^\flat$.

We have already seen how we can easily handle small magnetic potentials in the context of Thomas' argument, and hence $W^\flat$ does not pose any problems. However, the possibly quite large $W^\sharp$ necessitates a new idea. The simplest idea would be to conjugate the operator in question with some suitably chosen function (as was done in p. 32). However, this can only remove from W the gradient of some scalar function, and this is not sufficient to transform W into something small.

The idea of Sobolev's proof is to conjugate with suitably constructed invertible zeroeth order pseudodifferential operators. That is, for special values of k, parametrized by a positive real parameter τ, we will construct invertible zeroeth order parameter-dependent toroidal pseudodifferential operators A and B such that

$$\left(\Delta_k + W \cdot \nabla_k\right) A = B\Delta_k + \left(W^\flat \cdot \nabla_k\right) A + Q,$$

where $W^\flat$ is small in the $\mathscr{L}^\infty$-norm and Q is essentially a zeroeth order pseudodifferential operator, and hence similar to an electric potential.

The technical details will seem more natural when seen from the perspective of the following calculations: Let $A = \operatorname{Op}(\sigma_A)$ be the sought-for zeroeth order pseudodifferential operator. Then

$$\begin{aligned}\left(\Delta_k + W^\sharp \cdot \nabla_k\right) A = A\Delta_k &+ \operatorname{Op}\left(-4\pi i\,(\xi + k) \cdot \nabla\sigma_A + 2\pi W^\sharp \cdot (\xi + k)\,\sigma_A\right)\\ &+ \operatorname{Op}\left(-\Delta\sigma_A - iW^\sharp \cdot \nabla\sigma_A\right).\end{aligned}$$

Here the last term is essentially of zeroeth order and poses no problems. We would like to have the second term to vanish. However, in order to have genuine pseudodifferential operators, we will cleave another portion off the first order term involving $W^\sharp$. Namely, we will introduce a cut-off function ψ_τ such that we will have good relations between $\langle\xi\rangle_\tau$ and τ on its support, allowing us to prove the required symbol inequalities, and such that on the support of $1 - \psi_\tau$, the second order symbol dominates the first-order one, and the latter may therefore be included in the former.

That is, we will get

$$\begin{aligned}\left(\Delta_k + W^\sharp \cdot \nabla_k\right) A = \operatorname{Op}&\left(\sigma_A + \frac{2\pi\,(1 - \psi_\tau)\,W^\sharp \cdot (\xi + k)\,\sigma_A}{4\pi^2\,(\xi + k)^2}\right)\Delta_k\\ &+ \operatorname{Op}\left(-4\pi i\,(\xi + k) \cdot \nabla\sigma_A + 2\pi\psi_\tau W^\sharp \cdot (\xi + k)\,\sigma_A\right)\\ &+ \operatorname{Op}\left(-\Delta\sigma_A - iW^\sharp \cdot \nabla\sigma_A\right).\end{aligned}$$

Now the second term may be made to vanish simply by defining $\sigma_A = \exp \circ \varphi$, where φ is defined via its Fourier coefficients as follows:

$$\widehat{\varphi}(\eta, \xi) = \frac{\psi_\tau(\xi)\,\widehat{W^\sharp}(\eta) \cdot (\xi + k)}{2i\,(\xi + k) \cdot \eta}.$$

This is possible by a careful choice of the complex vector k.

From these considerations, it is reasonable to define the symbols σ_A, σ_B and σ_Q of A, B and Q, respectively,

$$\begin{cases} \sigma_A = \exp \circ \varphi, \\ \sigma_B = \sigma_A + \dfrac{(1-\psi_\tau)\, W^\sharp \cdot (\xi + k)\, \sigma_A}{2\pi\, (\xi+k)^2}, \quad \text{and} \\ \sigma_Q = -\Delta \sigma_A - i W^\sharp \cdot \nabla \sigma_A. \end{cases}$$

It will turn out that the corresponding operators A and B are indeed invertible zeroeth pseudodifferential operators and that Q is also of zeroeth order when multiplied by a suitable power of τ. Finally, the previously seen tools may be used to prove the estimate $(\mathscr{E})$ for the operator $B\Delta_k + \left(W^\flat \cdot \nabla_k\right) A + Q + VA$.

The set-up

The parameters

We first introduce the parameter values we will be using in the rest of this chapter. The intertwining operators, which will be parameter-dependent toroidal pseudodifferential operators as discussed in the previous chapter, will depend on a global parameter τ which takes values from the set

$$\Lambda = [1, \infty[\,.$$

Of course, ultimately we will work only with sufficiently large members of Λ.

We let $L \in \mathbb{Z}_+$ be smallest possible such that

$$\operatorname{supp} \widehat{W^\sharp} \subseteq \,]-L, L[^{\,n}\,.$$

The choice of k is a bit more involved. We define

$$v = \left\langle 1, L, L^2, \ldots, L^{n-1} \right\rangle \in \mathbb{Z}^n,$$

and the values of k will be as in the more general version of Thomas' argument (cf. p. 30), that is,

$$k = \frac{v}{2\,|v|^2} + \frac{i\tau v}{|v|}.$$

For this choice of v, it happens that $\left|v\right| \asymp L^{n-1}$, so that

$$\left\|u\right\|_{\mathscr{L}^2(\mathbb{T}^n)} \ll_{W^\sharp} \frac{1}{\tau} \left\|\Delta_k u\right\|_{\mathscr{L}^2(\mathbb{T}^n)},$$

for every $u \in \mathscr{H}^2(\mathbb{T}^n)$.

One nice consequence of this choice of v is simply that inner products $v \cdot \xi$ with integer vectors $\xi \in \mathbb{Z}^n \setminus \{0\}$ with small components are of size at least one. More precisely,

$$\left|v \cdot \xi\right| \geqslant 1$$

for every $\xi \in \mathbb{Z}^n \setminus \left\{0\right\}$ with each component being strictly smaller (in absolute value) than L. This is an easy number theoretical result and follows from a simple induction on n.

The cut-off function ψ_τ

Before we introduce the intertwining operators, we must define a certain cut-off function. Let $c \in \,]0,1[$ be a small constant. Let $\chi_1 \in \mathscr{C}_c^\infty(\mathbb{R})$ be a cut-off function such that $\chi_1 \equiv 1$ on $[1-c, 1+c]$, and that $\operatorname{supp}\chi_1 = \left[\frac{1-c}{2}, 2+2c\right]$. Let $\chi_2 \in \mathscr{C}_c^\infty(\mathbb{R})$ be another cut-off function such that $\chi_2 \equiv 1$ on $[-c, c]$, and that $\operatorname{supp}\chi_2 = [-2c, 2c]$. We then define the cut-off function $\psi_\tau \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ to be

$$\psi_\tau(\xi) = \chi_1\Big(\frac{|\xi + \Re k|}{\tau}\Big)\,\chi_2\Big(\frac{(\xi+\Re k)\cdot v}{|v|\,\tau}\Big)$$

for every $\xi \in \mathbb{R}^n$. The reason ψ_τ is not defined only on $\mathbb{Z}^n$ is that in order to prove estimates involving ψ_τ, it is much easier to consider ξ-derivatives than ξ-differences (see p. 43).

The cut-off function ψ_τ has been so defined that on the complement of its support, the symbol of the second order terms dominates that of the first-order terms. In loose words, the support of $1-\psi_\tau$ is the region into which Thomas' argument can reach without much effort. More precisely, we can make the following

Observation. *On the support of ψ_τ, we have*

$$(\xi+k)^2 \ll \tau^2 \quad \textit{and} \quad \langle\xi\rangle_\tau \asymp \tau.$$

Furthermore, on the support of $1-\psi_\tau$, we have $(\xi+k)^2 \gg \tau^2$. Consequently, on the supports of all the derivatives of ψ_τ and $1-\psi_\tau$, one can rely on the estimates

$$\langle\xi\rangle_\tau \asymp \tau \quad \textit{and} \quad (\xi+k)^2 \asymp \tau^2.$$

Proof. On the support of ψ_τ, we have

$$\xi^2 \asymp (\xi+\Re k)^2 \asymp \tau^2,$$

and therefore also $\langle\xi\rangle_\tau \asymp \tau$.

The real and imaginary parts of the symbol $(\xi+k)^2$ are

$$\Re\,(\xi+k)^2 = (\xi+\Re k)^2 - (\Im k)^2 = (\xi+\Re k)^2 - \tau^2,$$

and

$$\Im\,(\xi+k)^2 = 2\,(\xi+\Re k)\cdot\Im k = 2\,(\xi+\Re k)\cdot\frac{v\tau}{|v|},$$

respectively. The definition of ψ_τ immediately implies that both the real part and the imaginary part are $O(\tau^2)$. This gives the first estimate.

The third estimate requires considering a few separate cases. Let us restrict ourselves to the support of $1-\psi_\tau$. Where $(\xi+\Re k)\cdot\Im k \gg \tau^2$, there clearly $(\xi+k)^2 \gg \tau^2$. On the other hand, where $(\xi+\Re k)\cdot\Im k \ll \tau^2$, there the estimate $(\xi+\Re k)^2 \asymp \tau^2$ fails. When the first term of the real part of $(\xi+k)^2$ is smaller than $(1-c)^2\,\tau^2$, it is dwarfed by the second term. When the first term is greater than $(1+c)^2\,\tau^2$, it dominates the second term. The remaining claims are trivial.

Observation. *The restriction $\psi_\tau\big|_{\mathbb{Z}^n}$, considered in the obvious way as a function on $\mathbb{T}^n \times \mathbb{Z}^n \times \Lambda$, belongs to the class $S^0_{1,0,\tau}$.*

To obtain the claim, it is easiest to consider ψ_τ as a function defined on $\mathbb{T}^n \times \mathbb{R}^n \times \Lambda$ in the natural way and to prove that the derivative $\partial_\xi^\alpha \psi_\tau(\xi)$ satisfies

$$\partial_\xi^\alpha \psi_\tau \ll_\alpha \langle\xi\rangle_\tau^{-|\alpha|},$$

for all $\xi \in \mathbb{R}^n$ and for each multi-index α. In particular, there is no x-dependence to worry about. Since ψ_τ is certainly bounded, we may assume that $\alpha \neq 0$.

Now, the functions χ_1 and χ_2 are bounded, as are all their derivatives. Hence the main point is how the extra factors one gets from the inner functions behave. The extra factors from $\chi_1(\dots)$ are the more complicated ones. From one differentiation of $\chi_1(\dots)$ one gets an extra factor of the form

$$\cdot\frac{\xi_\ell}{\tau\,|\xi + \Re k|}.$$

The ∂_ξ^α-derivatives of this are of the form

$$\frac{1}{\tau} \cdot \frac{P_\alpha(\xi,\tau)}{|\xi + \Re k|^{2\left(|\alpha| + \frac{1}{2}\right)}},$$

where P_α is a polynomial of degree at most $|\alpha| + 1$. This form already makes it apparent why everything will work.

From the inner function of χ_2 one gets the simple extra factors of the form $\frac{v_\ell}{|v|\tau}$ which are clearly of the form $O\big(\tau^{-1}\big)$.

Finally, the claim follows by combining the above facts and using the estimates that hold on the support of the derivatives of ψ_τ.

Observation. *The function on $\mathbb{T}^n \times \mathbb{Z}^n$, defined by the expression*

$$\frac{1 - \psi_\tau(\xi)}{(\xi + k)^2},$$

where ξ ranges over $\mathbb{Z}^n$ and there is no x-dependence, is a symbol belonging to the class $S^{-2}_{1,0,\tau}$.

Denote the function in question temporarily by σ. Since there is no x-dependence, we only need to prove that $\partial_\xi^\alpha \sigma \ll_\alpha \langle\xi\rangle_\tau^{-2-|\alpha|}$ for all multi-indices α. For any multi-index β, the difference expression

$$\partial_\xi^\beta \frac{1}{(\xi + k)^2} \quad \text{has the form} \quad \frac{P_\beta(\xi,k)}{(\xi + k)^{2|\beta|+2}},$$

with some polynomial P_β of degree at most $|\beta|$. On the support of $1 - \psi_\tau$, this may clearly be estimated by $O\big(\langle\xi\rangle_\tau^{|\beta|-2|\beta|-2}\big) = O\big(\langle\xi\rangle_\tau^{-|\beta|-2}\big)$.

But now by Leibnitz' rule,

$$\begin{aligned}
\partial_\xi^\alpha \frac{1 - \psi_\tau}{(\xi + k)^2} &= \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \partial_\xi^{\alpha-\beta}\,(1 - \psi_\tau) \cdot \partial_\xi^\beta \frac{1}{(\xi + k)^2} \\
&\ll_\alpha \sum_{\beta \leqslant \alpha} \langle\xi\rangle_\tau^{-|\alpha-\beta|}\,\langle\xi\rangle_\tau^{-2-|\beta|} \ll_\alpha \langle\xi\rangle_\tau^{-2-|\alpha|}.
\end{aligned}$$

The intertwining operators

Our potential W consists of two parts, $W^\sharp$ and $W^\flat$. Of these, $W^\flat$ is small and therefore manageable by Thomas' approach. What we need to do is to make $W^\sharp$ go away, leaving only terms of order smaller than one. This will be achieved by transforming away a large proportion of $W^\sharp$ leaving first-order terms which can be incorporated into the second-order terms in a manner which saves Thomas' argument.

As mentioned before, we will construct parameter-dependent toroidal pseudodifferential operators A, B and Q with agreeable properties and satisfying the identity

$$\left(\Delta_k + W^\sharp \cdot \nabla_k\right) A = B\Delta_k + Q.$$

The operators themselves

We begin by defining a zeroeth order parameter-dependent pseudodifferential symbol φ, or rather its Fourier coefficients, via the formula

$$\widehat{\varphi}(\eta, \xi) \stackrel{\text{def}}{=} \frac{\psi_\tau(\xi)\, \widehat{W^\sharp}(\eta) \cdot (\xi + k)}{2i\,(\xi + k) \cdot \eta},$$

for all $\xi \in \mathbb{Z}^n$, and every $\eta \in \mathbb{Z}^n \setminus \{0\}$ with each component having absolute value strictly smaller than L. For other values of $\eta \in \mathbb{Z}^n$, we simply set $\widehat{\varphi}(\eta, \xi)$ equal to zero.

Observation. *The function φ is a genuine symbol from the class $S^0_{1,0,\tau}$.*

Proof. We shall consider φ as a function on $\mathbb{R}^n \times \mathbb{R}^n$, and prove that

$$\partial_\xi^\alpha \partial_x^\beta \varphi \ll_{\alpha\beta} \langle \xi \rangle_\tau^{-|\alpha|}$$

for all $x, \xi \in \mathbb{R}^n$ and for all multi-indices α and β. Here ∂_ξ^α refers to differentiation, not to taking differences.

We first observe that on the support of $\widehat{\varphi}$, we can estimate the denumerator as follows:

$$(\xi + k) \cdot \eta \gg \Im k \cdot \eta = \frac{\tau v \cdot \eta}{|v|} \gg \frac{\tau}{|v|} \asymp_L \tau.$$

Since $\tau \asymp \langle \xi \rangle_\tau$ on $\operatorname{supp} \varphi$, we get the following bound for φ:

$$\varphi(x, \xi) \ll \sum_{|\eta|_\infty < L} \left| \frac{\psi_\tau(\xi)\, \widehat{W^\sharp}(\eta) \cdot (\xi + k)}{2i\,(\xi + k) \cdot \eta} \right| \ll \sum_{|\eta|_\infty < L} \frac{\langle \xi \rangle_\tau}{\tau} \ll_{W^\sharp} 1.$$

Next we tackle differentiation with respect to ξ. Let α be an arbitrary multi-index. Fix some $\eta \in \mathbb{Z}^n \setminus \{0\}$ with $|\eta|_\infty < L$. Then the derivative of the inverse of the denominator of $\widehat{\varphi}$ will satisfy

$$\partial_\xi^\alpha \frac{1}{(\xi + k) \cdot \eta} = \frac{\eta^\alpha}{\left((\xi + k)\, \cdot \eta\right)^{|\alpha| + 1}} \ll_{W^\sharp} \frac{1}{\tau^{|\alpha| + 1}}.$$

Now, on the support of φ, the ∂_ξ^α-derivative of the Fourier coefficient $\widehat{\varphi}(\eta,\xi)$ satisfies the estimates

$$\begin{aligned}\partial_\xi^\alpha \widehat{\varphi}(\eta,\xi) &= \frac{1}{2i} \sum_{\beta\leqslant\alpha} \sum_{\gamma\leqslant\alpha-\beta} \binom{\alpha}{\beta}\binom{\alpha-\beta}{\gamma} \partial_\xi^{\alpha-\beta-\gamma}\psi_\tau(\xi) \\ &\qquad\qquad \cdot \partial_\xi^\beta \big(\widehat{W^\sharp}(\eta)\cdot(\xi+k)\big)\cdot \partial_\xi^\gamma \frac{1}{(\xi+k)\cdot\eta} \\ &\ll_\alpha \sum_{\beta\leqslant\alpha}\sum_{\gamma\leqslant\alpha-\beta} \langle\xi\rangle_\tau^{-|\alpha-\beta-\gamma|} \langle\xi\rangle_\tau^{1-|\beta|} \cdot \frac{1}{\langle\xi\rangle_\tau^{|\gamma|-1}} \ll_\alpha \langle\xi\rangle_\tau^{-|\alpha|}\,.\end{aligned}$$

Now we obtain $\partial_\xi^\alpha\varphi \ll_\alpha \langle\xi\rangle_\tau^{-|\alpha|}$ the same way we obtained $\varphi \ll 1$.

Our next observation is that differentiating $\partial_\xi^\alpha\varphi$ with respect to x, say by ∂_x^β for some multi-index β, gives essentially just an extra factor η^β to each Fourier coefficient of φ, and since this factor is in each case $\ll L^{|\beta|} \ll 1$, we conclude that φ belongs to $S^0_{1,0,\tau}$.

——:——

Since $\exp\circ\varphi \in S^0_{1,0,\tau}$ (cf. p. 42), we may define the operator A to be simply

$$A \overset{\text{def}}{=} \operatorname{Op}(\sigma_A) \overset{\text{def}}{=} \operatorname{Op}(\exp\circ\varphi)\,.$$

Observation. *The operator $A \in \operatorname{Op}\big[S^0_{1,0,\tau}\big]$ is invertible for sufficiently large $\tau \in \Lambda$.*

The function $e^{-\varphi}$ is also a symbol from the class $S^0_{1,0,\tau}$. Now, by considering the first term in the composition formula of pseudodifferential operators, we see that both of the compositions of A with $\operatorname{Op}(e^{-\varphi})$ are of the form $1+R$, where 1 denoting $\mathrm{id}_{\mathscr{L}^2(\mathbb{T}^n)}$ and R being an operator from the class $S^{-1}_{1,0,\tau}$.

But now the $\mathscr{L}^2$-boundedness result guarantees that the operator norm of R is bounded by $O\big(\tau^{-1}\big)$ which goes to zero as $\tau \longrightarrow \infty$. Consequently, both of the operators $1+R$ are invertible by Neumann series, and therefore also both of the compositions of A with $\operatorname{Op}(e^{-\varphi})$ are invertible with bounded inverses, for sufficiently large τ.

——:——

We define the symbol σ_B of the operator B to be

$$\sigma_B(x,\xi) \overset{\text{def}}{=} \sigma_A + \frac{\big(1-\psi_\tau(\xi)\big)\,W^\sharp\cdot(\xi+k)\,\sigma_A}{2\pi\,(\xi+k)^2}$$

for all $x\in\mathbb{T}^n$ and every $\xi\in\mathbb{Z}^n$ (compare with p. 63).

Observation. *The operator B belongs to the class $\operatorname{Op}\big[S^0_{1,0,\tau}\big]$ and is also invertible for sufficiently large values of τ.*

The second term in the definition of σ_B is a product of a symbol in $S^{-2}_{1,0,\tau}$ with a symbol from $S^{1}_{1,0,\tau}$ and another symbol from $S^{0}_{1,0,\tau}$, and hence belongs to $\sigma \in S^{-1}_{1,0,\tau}$. Hence it is clear that $\sigma_B \in S^{0}_{1,0,\tau}$, and the invertibility of B for large τ follows easily. Namely, the operator is a sum of the invertible operator A with an operator whose norm tends to zero as $\tau \longrightarrow \infty$.

——:——

Having introduced A and B, we finally introduce the remaining operator Q (compare to p. 63):

$$\sigma_Q \stackrel{\text{def}}{=} \operatorname{Op}\bigl(-\Delta\sigma_A - iW^\sharp \cdot \nabla\sigma_A\bigr)\,.$$

Observation. *The operator Q belongs to the class $\operatorname{Op}\bigl[S^{0}_{1,0,\tau}\bigr]$.*

This follows immediately from $\sigma_A \in S^{0}_{1,0,\tau}$.

The intertwining identity

We will now collect the above observations and state the intertwining identity in the form it will be used in the next section.

The Intertwining Identity. *For sufficiently large values of $\tau \in \Lambda$ the operators $A, B, Q \in \operatorname{Op}\bigl[S^{0}_{1,0,\tau}\bigr]$ extend to invertible bounded linear operators with bounded inverses of $\mathscr{L}^2(\mathbb{T}^n)$, and*

$$\bigl(\Delta_k + W^\sharp \cdot \nabla_k\bigr) \circ A = B \circ \Delta_k + Q.$$

The only remaining claim to check is the identity itself. This is achieved through straightforward computation.

From the identities (cf. p. 48)

$$\frac{\partial}{\partial x_\ell} \circ A = A \circ \frac{\partial}{\partial x_\ell} + \operatorname{Op}\Bigl(\frac{\partial\sigma_A}{\partial x_\ell}\Bigr) = \operatorname{Op}\Bigl(2\pi i\xi_\ell\sigma_A + \frac{\partial\sigma_A}{\partial x_\ell}\Bigr)\,,$$

it follows that

$$\bigl(W^\sharp \cdot \nabla_k\bigr)\, A = W^\sharp \cdot (-i\nabla + 2\pi k)\, A = \operatorname{Op}\bigl(2\pi W^\sharp \cdot (\xi + k)\,\sigma_A - iW^\sharp \cdot \nabla\sigma_A\bigr)\,.$$

In the same vein,

$$\begin{aligned}
\bigl(\Delta_k + W^\sharp \cdot \nabla_k\bigr)\, A = \operatorname{Op}\bigl(&4\pi^2\,(\xi + k)^2\,\sigma_A - 4\pi i\,(\xi + k)\cdot\nabla\sigma_A \\
&- \Delta\sigma_A + 2\pi W^\sharp \cdot (\xi + k)\,\sigma_A - iW^\sharp \cdot \nabla\sigma_A\bigr) \\
= A\Delta_k &+ \operatorname{Op}\bigl(-4\pi i\,(\xi + k)\cdot\nabla\sigma_A + 2\pi W^\sharp \cdot (\xi + k)\,\sigma_A\bigr) + Q \\
= A\Delta_k &+ \operatorname{Op}\Bigl(2\pi\,\bigl(1 - \psi_\tau(\xi)\bigr)\,W^\sharp \cdot (\xi + k)\,\sigma_A\Bigr) + Q \\
= B\Delta_k &+ Q,
\end{aligned}$$

as required.

The $\mathscr{L}^2$-estimates

We will now combine all the previous considerations to obtain the required contradiction. Let the function $u \in \mathscr{H}^2(\mathbb{T}^n)$ be arbitrary, and denote the $\mathscr{L}^2(\mathbb{T}^n)$-function $(\Delta_k + W \cdot \nabla_k + V)\, u$ by f. We want to show that

$$\|u\|_{\mathscr{L}^2(\mathbb{T}^n)} \ll \frac{1}{\tau}\|f\|_{\mathscr{L}^2(\mathbb{T}^n)},$$

for large enough $\tau \in \Lambda$.

We have gathered many intermediate norm estimates. We know (from p. 58) that

$$\begin{cases} \|\Delta_k^{-1}\|_{\mathscr{L}^2(\mathbb{T}^n)\longrightarrow\mathscr{L}^2(\mathbb{T}^n)} \ll \frac{|v|}{\tau} \ll \frac{1}{\tau}, \quad \text{and that} \\ \|\nabla_k\Delta_k^{-1}\|_{\mathscr{L}^2(\mathbb{T}^n)\longrightarrow\mathscr{L}^2(\mathbb{T}^n;\,\mathbb{C}^n)} \ll |v| \ll 1. \end{cases}$$

We also have bounded invertible operators A and B and a bounded operator Q such that

$$\left(\Delta_k + W^\sharp \cdot \nabla_k\right) A = B\Delta_k + Q,$$

where the operators A, B and Q satisfy the norm estimates

$$\begin{cases} \|A^{\pm 1}\|_{\mathscr{L}^2(\mathbb{T}^n)\longrightarrow\mathscr{L}^2(\mathbb{T}^n)} \ll 1, \\ \|B^{\pm 1}\|_{\mathscr{L}^2(\mathbb{T}^n)\longrightarrow\mathscr{L}^2(\mathbb{T}^n)} \ll 1, \quad \text{and} \\ \|Q\|_{\mathscr{L}^2(\mathbb{T}^n)\longrightarrow\mathscr{L}^2(\mathbb{T}^n)} \ll 1. \end{cases}$$

Furthermore, $\mathrm{Op}(-i\nabla\sigma_A) \in \mathrm{Op}\big[S^0_{1,0,\tau}\big]$, so that

$$\|\mathrm{Op}(-i\nabla\sigma_A)\|_{\mathscr{L}^2(\mathbb{T}^n)\longrightarrow\mathscr{L}^2(\mathbb{T}^n)} \ll 1.$$

We define $\widetilde{u} = B\Delta_k A^{-1} u \in \mathscr{L}^2(\mathbb{T}^n)$, so that

$$\|u\|_{\mathscr{L}^2(\mathbb{T}^n)} \ll \|\Delta_k^{-1}\|_{\mathscr{L}^2(\mathbb{T}^n)\longrightarrow\mathscr{L}^2(\mathbb{T}^n)}\|\widetilde{u}\|_{\mathscr{L}^2(\mathbb{T}^n)} \ll \frac{1}{\tau}\|\widetilde{u}\|_{\mathscr{L}^2(\mathbb{T}^n)},$$

and

$$\begin{aligned} f &= (\Delta_k + W \cdot \nabla_k + V)\, u \\ &= \Big(\Delta_k + W^\sharp \cdot \nabla_k + W^\flat \cdot \nabla_k + V\Big)\, A\,(\Delta_k)^{-1}\, B^{-1}\widetilde{u} \\ &= \Big(1 + Q\,(\Delta_k)^{-1}\, B^{-1} + W^\flat \cdot \nabla_k A\,(\Delta_k)^{-1}\, B^{-1} + VA\,(\Delta_k)^{-1}\, B^{-1}\Big)\widetilde{u}. \end{aligned}$$

Denote the operator $(1 + \ldots)$ by Υ. We will prove that for sufficiently large values of τ, the operator Υ is invertible with operator norm $\|\Upsilon^{\pm 1}\| \asymp 1$. To prove this, it suffices to show that for large τ the three last operator terms of $(1 + \ldots)$ are sufficiently small, say each smaller than $\frac{1}{4}$, for then Υ is invertible by Neumann series and $\|\Upsilon^{\pm 1}\| \leqslant 4$.

First, we have

$$\|Q\Delta_k^{-1}B^{-1}\|_{\mathscr{L}^2(\mathbb{T}^n)\longrightarrow\mathscr{L}^2(\mathbb{T}^n)} \ll \frac{1}{\tau} \xrightarrow[\tau\longrightarrow\infty]{} 0.$$

Similarly,

$$\|VA\Delta_k^{-1}B^{-1}\|_{\mathscr{L}^2(\mathbb{T}^n)\longrightarrow\mathscr{L}^2(\mathbb{T}^n)} \ll \frac{1}{\tau} \xrightarrow[\tau\longrightarrow\infty]{} 0.$$

Finally, since $\nabla_k A = A\nabla_k + \mathrm{Op}(-i\nabla\sigma_A)$, we can estimate

$$\begin{aligned}
&\left\|W^\flat \cdot \nabla_k A \Delta_k^{-1} B^{-1}\right\|_{\mathscr{L}^2(\mathbb{T}^n)\longrightarrow\mathscr{L}^2(\mathbb{T}^n)} \\
&\qquad\qquad \ll \left\|W^\flat\right\|_{\mathscr{L}^\infty(\mathbb{T}^n;\mathbb{C}^n)} \Big(\left\|A\nabla_k \Delta_k^{-1} B^{-1}\right\|_{\mathscr{L}^2(\mathbb{T}^n)\longrightarrow\mathscr{L}^2(\mathbb{T}^n;\mathbb{C}^n)} \\
&\qquad\qquad\qquad + \left\|\mathrm{Op}(-i\nabla\sigma_A)\, \Delta_k^{-1} B^{-1}\right\|_{\mathscr{L}^2(\mathbb{T}^n)\longrightarrow\mathscr{L}^2(\mathbb{T}^n;\mathbb{C}^n)} \Big) \\
&\qquad\qquad \ll \left\|W^\flat\right\|_{\mathscr{L}^\infty(\mathbb{T}^n;\mathbb{C}^n)} \left(1 + \frac{1}{\tau}\right),
\end{aligned}$$

and so the term $W^\flat \cdot \nabla_k A \Delta_k^{-1} B^{-1}$ may be made as small as required simply by taking $\left\|W^\flat\right\|_{\mathscr{L}^\infty(\mathbb{T}^n;\mathbb{C}^n)} \ll 1$, where the implicit constant depends heavily on $W^\sharp$. We conclude that $\left\|\Upsilon^{\pm 1}\right\|_{\mathscr{L}^2(\mathbb{T}^n)\longrightarrow\mathscr{L}^2(\mathbb{T}^n)} \asymp 1$, so that $\left\|f\right\|_{\mathscr{L}^2(\mathbb{T}^n)} \asymp \left\|\widetilde{u}\right\|_{\mathscr{L}^2(\mathbb{T}^n)}$, and

$$\left\|u\right\|_{\mathscr{L}^2(\mathbb{T}^n)} \ll \frac{1}{\tau}\left\|\widetilde{u}\right\|_{\mathscr{L}^2(\mathbb{T}^n)} \ll \frac{1}{\tau}\left\|f\right\|_{\mathscr{L}^2(\mathbb{T}^n)}. \qquad \text{Q.E.D.}$$

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