

Resonances and Ω -results for Exponential Sums Related to Maass Forms for $SL(n, \mathbb{Z})$

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Abstract

We obtain resonances for short exponential sums weighted by Fourier coefficients of Maass forms for $SL(n, \mathbb{Z})$. This involves deriving asymptotics for the integrals appearing in the $GL(n)$ Voronoi summation formula. As an application, we also prove an Ω -result for short sums of Fourier coefficients.

1 Introduction

1.1 Exponential sums related to cusp forms

Little is known about Fourier coefficients of holomorphic cusp forms or Maass forms both in $GL(2)$ and in $GL(n)$ for $n > 2$. Therefore, it makes sense to study exponential sums

$$\sum_{M \leq m \leq M+\Delta} a(m) e(m\alpha) \tag{1}$$

of the Fourier coefficients of cusp forms with $M \in [1, \infty[$ and $\Delta \in [1, M]$. Instead of the function above, one might want to have some other function in the place of the function αm , say, the function αm^β . However, in what follows, we concentrate on so called linear exponential sums, that is, sums of the type (1) with $\alpha \in \mathbb{R}$. Furthermore, we may assume that $0 \leq \alpha < 1$.

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In particular, short sums are interesting for studying properties of Fourier coefficients, and they are a natural analogue for studying classical number-theoretic error terms in short intervals.

The behaviour of the exponential sums is extremely intriguing: One might expect square-root cancellation, and while this is not an unreasonable assumption in the $GL(2)$ setting when $\Delta \ll M^{1/2}$, Ernvall-Hytönen [4] has proved that

$$\begin{aligned} & \sum_{M \leq m \leq M+\Delta} a(m) e\left(\frac{m\sqrt{d}}{\sqrt{M}}\right) w(m) \\ &= C a(d) d^{-1/4} \int_M^{M+\Delta} x^{-1/4} w(x) e\left(\frac{x\sqrt{d}}{\sqrt{M}} - 2\sqrt{dx}\right) dx + O(1), \end{aligned} \quad (2)$$

for the normalised Fourier coefficients $a(n)$ of a holomorphic cusp form

$$\sum_{n=1}^{\infty} a(n) n^{(\kappa-1)/2} e(nz)$$

of weight $\kappa \in \mathbb{Z}_+$ for $SL(2, \mathbb{Z})$, where $d \in \mathbb{Z}_+$ is fixed and C is a non-zero constant only depending on the weight of the underlying holomorphic cusp form, and w is a suitable weight function supported on $[M, M + \Delta]$. Since the integrand does not oscillate, the integral is $\asymp \Delta M^{-1/4}$. This shows that for $M^{1/2+\varepsilon} \ll \Delta \leq M^{3/4} d^{-1/2}$ the size of the sum is $\asymp \Delta M^{-1/4}$ unless $a(d) = 0$, in which case the size of the sum is $O(1)$. The estimate $\Delta M^{-1/4}$ is not only larger than square root size, but also grows linearly with Δ , which is quite surprising. A similar phenomenon has been observed for $GL(2)$ and $GL(3)$ Maass forms by Ernvall-Hytönen [6, 5], and for non-linear sums by Iwaniec, Luo and Sarnak [18] and by Ren and Ye [34].

Ω -results are the other side of the story. With an Ω -result we mean a lower bound in the following sense:

$$\sum_{M \leq m \leq M+\Delta} a(m) = \Omega(F(M, \Delta))$$

means that

$$\sum_{M \leq m \leq M+\Delta} a(m) \neq o(F(M, \Delta)).$$

The result above (2) has been used to prove the following Ω -result:

$$\sum_{M \leq m \leq M+cM^{1/2}} a(m) = \Omega(M^{1/4}),$$

where $c \in \mathbb{R}_+$ is an arbitrary fixed coefficient. This Ω -result extends the earlier work by Ivić [17] (see also [21]), where he has shown that

$$\sum_{M \leq m \leq M+\Delta} a(m) = \Omega(\sqrt{\Delta})$$

for $\Delta = o(\sqrt{M})$.

While upper and lower bounds are relatively widely studied in the $\mathrm{GL}(2)$ setting (see e.g. [37, 22, 3, 14, 15]), even though there are plenty of open questions left, the situation in $\mathrm{GL}(n)$ for $n > 2$ is much less understood. The best uniform upper bound for exponential sums for $\mathrm{GL}(3)$ is the one due to Miller [29],

$$\sum_{m \leq M} A(m, 1) e(m\alpha) \ll M^{3/4+\varepsilon},$$

which holds uniformly in $\alpha \in \mathbb{R}$. Ren and Ye [35] studied how the bound can be improved in the presence of a weight function in terms of rational approximations of α . The dependence on weight and/or spectral parameters has been studied in both $\mathrm{GL}(2)$ and in $\mathrm{GL}(3)$ [27, 9].

1.2 What we do in this paper

Let ψ be a Maass form for $\mathrm{SL}(n, \mathbb{Z})$, where $n \geq 2$ (for a general reference, see [10]). It comes with an attached L -function called the Godement–Jacquet L -function:

$$L(s) = \sum_{m=1}^{\infty} \frac{A(m, 1, \dots, 1)}{m^s}.$$

The Fourier coefficients $A(m, 1, \dots, 1)$ satisfy a pointwise bound

$$A(m, 1, \dots, 1) \ll m^{\vartheta+\varepsilon}$$

for every $m \in \mathbb{Z}_+$ for some $\vartheta \in [0, \infty[$ depending on n . It is known that we can choose $\vartheta = 0$ for holomorphic cusp forms [2], $\vartheta = \frac{7}{64}$ for Maass forms in $\mathrm{GL}(2)$, $\vartheta = \frac{5}{14}$ for Maass forms in $\mathrm{GL}(3)$ and $\vartheta = \frac{9}{22}$ for Maass forms in $\mathrm{GL}(4)$ [24]. For $\mathrm{GL}(n)$ Maass forms with $n \geq 5$, we can choose $\vartheta = \frac{1}{2} - \frac{1}{n^2+1}$ [28]. It has been conjectured that $\vartheta = 0$ is admissible. Also, from the study of the Rankin–Selberg convolutions, it follows that

$$\sum_{m \leq x} |A(m, 1, \dots, 1)|^2 \ll x$$

for $x \in \mathbb{R}_+$ (see e.g. Sect. 12.1. in [10]).

Our object here is to study certain resonances for short exponential sums involving the coefficients $A(m, 1, \dots, 1)$. This will lead to an Ω -result for sums of coefficients.

We start by considering the Voronoi type summation formula for exponential sums involving Fourier coefficients of cusp forms. Namely, we derive asymptotics for the integrals in the Voronoi type summation formula for $GL(n)$ Maass forms. The details can be found in Section 5.

We then use these asymptotics to prove the following resonance result:

Theorem 1. *Let $M^{1-1/n+\varepsilon} \ll \Delta \ll M$, and let d be a fixed positive integer. Also, let $w \in C_c^\infty(\mathbb{R}_+)$ be supported in the interval $[M, M + \Delta]$ with $w^{(\nu)}(x) \ll_\nu \Delta^{-\nu}$ for $\nu \in \mathbb{Z}_+ \cup \{0\}$. Then*

$$\begin{aligned} \sum_{M \leq m \leq M+\Delta} A(m, 1, \dots, 1) w(m) e\left(\frac{d^{1/n} m}{M^{1-1/n}}\right) &= \frac{A(1, \dots, 1, d)}{d^{1/2-1/(2n)} \sqrt{n}} e\left(-\frac{n+3}{8}\right) \\ &\cdot \int_M^{M+\Delta} w(x) e\left(\frac{d^{1/n} x}{M^{1-1/n}} - n x^{1/n} d^{1/n}\right) x^{1/(2n)-1/2} dx \\ &+ O(\Delta M^{-1/2-1/(2n)}). \end{aligned}$$

This is a generalization of Theorem 1.1 from [6], and a generalization of the corresponding results in the $GL(2)$ setting. Notice that in Theorem 1.1 in [6], the constant $\frac{-1}{\pi d^{-1/3}}$ appears accidentally in front of the main term. It should not be there, and once removed, the result agrees with this theorem.

When $\Delta \ll_{n,d} M^{1-1/(2n)}$, the integrand on the right-hand side in Theorem 1 does not oscillate and the integral must be $\asymp \Delta M^{1/(2n)-1/2}$, and so we get the following corollary.

Corollary 2. *Let $M^{1-1/n+\varepsilon} \ll \Delta \ll_{n,d} M^{1-1/(2n)}$, where the last implicit constant is sufficiently small, let $d \in \mathbb{Z}_+$ be fixed, and let $w \in C_c^\infty(\mathbb{R}_+)$ be supported in $[M, M + \Delta]$ and satisfy $w^{(\nu)}(x) \ll_\nu \Delta^{-\nu}$ for every $\nu \in \mathbb{Z}_+ \cup \{0\}$. Then, if $A(1, \dots, 1, d) \neq 0$, we have*

$$\sum_{M \leq m \leq M+\Delta} A(m, 1, \dots, 1) e\left(\frac{d^{1/n} m}{M^{1-1/n}}\right) w(m) \asymp \Delta M^{1/(2n)-1/2}.$$

In Corollaries 14 and 15 below, the weight function is removed. In particular, we get

Corollary 3. *Let $d \in \mathbb{Z}_+$ be fixed, and assume that $A(1, \dots, 1, d) \neq 0$. Then, for $M^{1-1/n+\varepsilon} \ll \Delta \ll M^{1-1/(2n)}$, the sum*

$$\sum_{M \leq m \leq M+\Delta} A(m, 1, \dots, 1) e\left(\frac{d^{1/n} m}{M^{1-1/n}}\right)$$

is $\Omega(\Delta M^{1/(2n)-1/2})$, and for $M^{1-1/(2n)} \ll \Delta \ll M$, the sum is $\Omega(M^{1/2})$.

For holomorphic cusp forms, numerical computations [8] suggest that the analogous result might also give the “right” upper bound, i.e. the linear sum (1) should always be $\ll \Delta M^{\varepsilon-1/4}$ in the length range $M^{1/2} \ll \Delta \ll M^{3/4}$. This has indeed been confirmed [7] in the range $M^{23/32} \ll \Delta \ll M^{3/4}$.

One can also prove the following non-linear resonance result:

Theorem 4. *Let $M^{1-1/n+\varepsilon} \ll \Delta \ll M$ and let d be a fixed positive integer. Also, let $w \in C_c^\infty(\mathbb{R}_+)$ be supported on the interval $[M, M + \Delta]$ with $w^{(\nu)}(x) \ll_\nu \Delta^{-\nu}$ for $\nu \in \mathbb{Z}_+ \cup \{0\}$. Then*

$$\sum_{M \leq m \leq M+\Delta} A(m, 1, \dots, 1) w(m) e(n d^{1/n} m^{1/n}) = \frac{A(1, \dots, 1, d)}{d^{1/2-1/(2n)} \sqrt{n}} e\left(-\frac{n+3}{8}\right) \cdot \int_M^{M+\Delta} w(x) x^{1/(2n)-1/2} dx + O(\Delta M^{-1/2-1/(2n)}).$$

While this paper was under review, it came to our attention that Theorem 4 has been obtained independently by Ren and Ye [36], who also independently performed arguments similar to those in Section 5 below.

We will use Theorem 1 to prove the following Ω -result:

Theorem 5. *Assume that $\Delta = o(M^{1/2-\vartheta-\varepsilon})$. Then*

$$\sum_{M \leq m \leq M+\Delta} A(m, 1, \dots, 1) = \Omega(M^{1/(2n)-1/2} \Delta).$$

Finally, it is interesting that one gets the following corollary concerning the determination of all the coefficients $A(m_1, \dots, m_{n-1})$ from a subset of coefficients.

Corollary 6. *Let M_1, M_2, \dots be a sequence of positive real numbers tending to infinity, and let ε be an arbitrarily small positive real number. Write*

$$I = \mathbb{Z} \cap \bigcup_{\ell=1}^{\infty} \left[M_\ell, M_\ell + M_\ell^{1-1/n+\varepsilon} \right].$$

Then the Fourier coefficients $A(m, 1, \dots, 1)$ with $m \in I$ uniquely determine all the Fourier coefficients $A(m_1, \dots, m_{n-1})$.

Indeed, the coefficients $A(m, 1, \dots, 1)$ with $m \in I$ uniquely determine an infinite sequence of exponential sums of the form appearing in Theorem 1, and taking $M \rightarrow \infty$ we can recover the coefficient $A(1, \dots, 1, d)$ for every $d \in \mathbb{Z}$. The strong multiplicity one theorem of Jacquet and Shalika [19]

(Theorem 12.6.1 in [10]) implies that the coefficients $A(1, \dots, 1, p)$ for primes p determine all the rest of the coefficients. Corollary 6 could perhaps be considered a relative of multiplicity one theorems (see e.g. the ones in [33, 1]).

2 Notation

The symbols \ll , \gg , \asymp , O and o are used for the usual asymptotic notation: for complex valued functions f and g in some set X , the notation $f \ll g$ means that $|f(x)| \leq C |g(x)|$ for all $x \in X$ for some implicit constant $C \in \mathbb{R}_+$. When the implied constant depends on some parameters α, β, \dots , we use $\ll_{\alpha, \beta, \dots}$ instead of mere \ll . The notation $g \gg f$ means $f \ll g$, and $f \asymp g$ means $f \ll g \ll f$. When f and g depend on M , we say that $f(M) = o(g(M))$ if $g(M)$ never vanishes and $f(M)/g(M) \rightarrow 0$ as $M \rightarrow \infty$.

All the implicit constants are allowed to depend on the underlying Maass form (hence also on n), on d , when it appears, and on ε , which denotes an arbitrarily small fixed positive number, which may not be the same on each occurrence. The implicit constants are also allowed to depend on a weight function w when such a function is used.

As usual, complex variables are written in the form $s = \sigma + it$, and we write $e(x)$ for $e^{2\pi i x}$. The subscript in the integral $\int_{(\sigma)}$ means that we integrate over the vertical line $\Re s = \sigma$. For simplicity, we write $\langle \cdot \rangle$ for $(1 + |\cdot|^2)^{1/2}$.

3 Basic properties of the Godement–Jacquet L -function

The Godement–Jacquet L -function attached to a Maass form ψ for $\mathrm{SL}(n, \mathbb{Z})$ with Fourier coefficients $A(m_1, m_2, \dots, m_{n-1})$ is the Dirichlet series

$$L(s) = \sum_{m=1}^{\infty} \frac{A(m, 1, \dots, 1)}{m^s}.$$

This converges absolutely for $\sigma > 1$ by the Rankin–Selberg estimate

$$\sum_{m \leq x} |A(m, 1, \dots, 1)|^2 \ll x.$$

(For this, see e.g. [10], Remark 12.1.8.) The Godement–Jacquet L -function has an entire analytic continuation and satisfies the functional equation

$$\pi^{-ns/2} G(s) L(s) = \pi^{-n(1-s)/2} \tilde{G}(1-s) \tilde{L}(1-s),$$

where \tilde{L} is the Godement–Jacquet L -function of the dual form of ψ , and given by the Dirichlet series

$$\tilde{L}(s) = \sum_{m=1}^{\infty} \frac{A(1, \dots, 1, m)}{m^s}$$

for $\sigma > 1$, and where

$$G(s) = \prod_{\ell=1}^n \Gamma\left(\frac{s - \lambda_{\ell}}{2}\right) \quad \text{and} \quad \tilde{G}(s) = \prod_{\ell=1}^n \Gamma\left(\frac{s - \tilde{\lambda}_{\ell}}{2}\right),$$

where λ_{ℓ} and $\tilde{\lambda}_{\ell}$ are certain complex parameters of ψ and $\tilde{\psi}$ with $\sum_{\ell=1}^n \lambda_{\ell} = \sum_{\ell=1}^n \tilde{\lambda}_{\ell} = 0$. It is known that $\Re \lambda_{\ell} \leq \frac{1}{2}$ and $\Re \tilde{\lambda}_{\ell} \leq \frac{1}{2}$ for each ℓ .

An elementary application of Stirling's formula says that when s lies in the vertical strips below and has a sufficiently large imaginary part (say, $|t| \geq 1$), the multiple Γ -factors can be replaced by a single quotient of two Γ -factors:

$$\frac{\tilde{G}(1-s)}{G(s)} = n^{ns-n/2} \frac{\Gamma\left(\frac{1-ns}{2}\right)}{\Gamma\left(\frac{ns-(n-1)}{2}\right)} (1 + O(|s|^{-1})),$$

as will be seen in Section 5.

The L -function has the usual growth properties in vertical strips. Since the L -function is bounded on the line $\sigma = 1 + \delta$, the functional equation and Stirling's formula (see Theorem 8 below) imply that $L(s) \ll \langle t \rangle^{n/2+n\delta}$ on the line $\sigma = -\delta$. Thus, by the Phragmén–Lindelöf principle for vertical strips,

$$L(s) \ll \langle t \rangle^{(1+\delta-\sigma)n/2}$$

in the vertical strip $-\delta \leq \sigma \leq 1 + \delta$.

4 Useful results

A Voronoi summation formula was implemented for $GL(n)$ in [30, 31, 11, 12]. The full formula involving additive twists is complicated but we only need the twistless formula, which is the following. It is the special case $q = 1$ of Theorem 1.1 in [12] or Theorem 1.10 in [31].

Theorem 7. *Let $f \in C_c^{\infty}(\mathbb{R}_+)$. Then*

$$\begin{aligned} & \sum_{m=1}^{\infty} A(m, 1, \dots, 1) f(m) \\ &= \sum_{m=1}^{\infty} \frac{A(1, \dots, 1, m)}{m} \frac{1}{2\pi i} \int_{(-\sigma_0)} \tilde{f}(s) \pi^{-n/2} \frac{\tilde{G}(1-s)}{G(s)} (\pi^n m)^s ds, \end{aligned}$$

where

$$\tilde{f}(s) = \int_0^{\infty} f(x) x^{s-1} dx$$

is the Mellin transform of f , and σ_0 is a large positive real number, depending on the form.

We need to be able to simplify and understand the behaviour of the Γ -factors appearing in the above integrals, and for this purpose Stirling's formula is needed. The following version can be found in e.g. Section 8.4.2 of [32].

Theorem 8. Fix some $K \in \mathbb{Z}_+$, $\delta \in]0, \pi[$ and $R \in \mathbb{R}_+$. Then

$$\Gamma(s) = \sqrt{2\pi} \exp\left(\left(s - \frac{1}{2}\right) \log s - s\right) \left(1 + \sum_{k=1}^K \frac{a_k}{s^k} + O_{K,\delta,R}(|s|^{-K-1})\right)$$

for all $s \in \mathbb{C}$ with $|s| \geq R$ and $|\arg s| \leq \pi - \delta$. The constant coefficients a_k do not depend on K , δ or R .

For vertical strips, $|\Gamma(s)|$ has the following useful estimate: given fixed real numbers $A < B$, we have for s with $A \leq \sigma \leq B$ and $t \geq 1$ that

$$|\Gamma(s)| = \sqrt{2\pi} t^{\sigma-1/2} e^{-\pi t/2} (1 + O(t^{-1})).$$

We will need a lemma for estimating exponential integrals. The following is Lemma 6 in [23].

Lemma 9. Let $a, b \in \mathbb{R}_+$ and $a < b$, and let $g \in C_c^\infty(\mathbb{R}_+)$ with $\text{supp } g \subseteq [a, b]$, and let G_0 and G_1 be such that

$$g^{(\nu)}(x) \ll_{\nu} G_0 G_1^{-\nu}$$

for all $x \in \mathbb{R}_+$ for each nonnegative integer ν . Also, let f be a holomorphic function defined in $D \subseteq \mathbb{C}$, which consists of all points in the complex plane with distance smaller than $\rho \in \mathbb{R}_+$ from the interval $[a, b]$ of the real axis. Assume that f is real-valued on $[a, b]$ and let $F_1 \in \mathbb{R}_+$ be such that

$$F_1 \ll |f'(z)|$$

for all $z \in D$. Then, for all positive integers P ,

$$\int_a^b g(x) e(f(x)) dx \ll_P G_0 (G_1 F_1)^{-P} \left(1 + \frac{G_1}{\rho}\right)^P (b - a).$$

5 Voronoi summation for Maass forms for $\mathrm{SL}(n, \mathbb{Z})$

The goal is to derive asymptotics for the integrals appearing in Theorem 7. The relevant integral is

$$\Omega(y) = \frac{1}{2\pi i} \int_{(-\sigma_0)}^{\infty} \int_0^{\infty} f(x) x^{s-1} dx \pi^{-n/2} \frac{\tilde{G}(1-s)}{G(s)} y^s ds,$$

where $f \in C_c^\infty(\mathbb{R}_+)$. The x -integral is certainly $\ll_N t^{-N}$, and so the s -integral behaves well.

Theorem 10. *For any $K \in \mathbb{Z}_+$ and $y \gg 1$, we have*

$$\Omega(y) = y^{1/2+1/(2n)} \int_0^{\infty} f(x) x^{1/(2n)-1/2} \mathcal{K}(x, y) dx,$$

where the kernel $\mathcal{K}(x, y)$ has, for $x \gg 1$ and $y \gg 1$, the asymptotics

$$\begin{aligned} \mathcal{K}(x, y) = \sum_{\ell=0}^K (xy)^{-\ell/n} (c_\ell^+ e(\pi^{-1} n x^{1/n} y^{1/n}) + c_\ell^- e(-\pi^{-1} n x^{1/n} y^{1/n})) \\ + O((xy)^{-(K+1)/n}). \end{aligned}$$

We emphasize that the implicit constant in the O -term is independent of f . Here the leading coefficients c_0^\pm are given by

$$c_0^\pm = \pi^{-(n+1)/2} \frac{1}{\sqrt{n}} e\left(\pm \frac{n+3}{8}\right).$$

Proof. The proof is based on the derivation of the asymptotics for integrals appearing in the ternary divisor function Voronoi summation formula in [16], also performed for $\mathrm{GL}(3)$ Maass forms in [26]. Fundamentally the idea in the proof is to first replace the unwieldy Γ -factors by a simple quotient of two Γ -factors, after which the resulting integrals then lead to integrals with shifted Γ -factors which will give the main terms of the asymptotics. The same idea has been implemented in Section 4 of [29], which we also recommend for further references on the approach. The proof below is, however, self-contained, and differs from the treatment in [29] in that the final integrals are evaluated in terms of J -Bessel functions whose asymptotics then give Theorem 10.

We shall pick $\Lambda \in \mathbb{R}_+$ which is larger than the abscissa σ_0 in the original integral, larger in absolute values than any of the λ_ℓ and $\tilde{\lambda}_\ell$, and larger than

$K/n + 1/(2n) - 1/2$. Restricting to s with $|s| \geq \Lambda$ and applying Stirling's formula, we get, after some simplification,

$$\begin{aligned} & \prod_{\ell=1}^n \frac{\Gamma\left(\frac{1-s-\tilde{\lambda}_\ell(\nu)}{2}\right)}{\Gamma\left(\frac{s-\lambda_\ell(\nu)}{2}\right)} \\ &= \frac{\exp\left(n\frac{1-s-1}{2} \log \frac{1-s}{2} - n\frac{1-s}{2} + n \sum_{k=1}^K \frac{a'_k 2^k}{(1-s)^k}\right)}{\exp\left(n\frac{s-1}{2} \log \frac{s}{2} - n\frac{s}{2} + n \sum_{k=1}^K \frac{a''_k 2^k}{s^k}\right)} \left(1 + O(|s|^{-K-1})\right). \end{aligned}$$

Here and below the implicit constants and the constant coefficients a'_k and a''_k will depend on λ_ℓ and $\tilde{\lambda}_\ell$.

It is not hard to guess from this that if we want to replace these gamma factors by a quotient of two gamma functions, the new quotient should be

$$\frac{\Gamma\left(\frac{1-ns}{2}\right)}{\Gamma\left(\frac{ns-(n-1)}{2}\right)}$$

Some careful calculation reveals that the quotient of the old gamma factors and the new gamma factors is

$$\prod_{\ell=1}^n \frac{\Gamma\left(\frac{1-s-\tilde{\lambda}_\ell(\nu)}{2}\right)}{\Gamma\left(\frac{s-\lambda_\ell(\nu)}{2}\right)} \cdot \frac{\Gamma\left(\frac{ns-(n-1)}{2}\right)}{\Gamma\left(\frac{1-ns}{2}\right)} = n^{ns-n/2} \left(1 + \sum_{k=1}^K \frac{C_k}{s^k} + O(|s|^{-K-1})\right).$$

We write

$$\begin{aligned} H(s) &= n^{-ns+n/2} \prod_{\ell=1}^n \frac{\Gamma\left(\frac{1-s-\tilde{\lambda}_\ell(\nu)}{2}\right)}{\Gamma\left(\frac{s-\lambda_\ell(\nu)}{2}\right)} \cdot \frac{\Gamma\left(\frac{ns-(n-1)}{2}\right)}{\Gamma\left(\frac{1-ns}{2}\right)} - 1 \\ &= \sum_{k=1}^K \frac{C_k}{s^k} + O(|s|^{-K-1}), \end{aligned}$$

and split the integral $\Omega(y)$ into two parts using $H(s)$:

$$\begin{aligned} \Omega(y) &= \frac{1}{2\pi i} \int_{(-\sigma_0)} \int_0^\infty f(x) x^{s-1} dx \pi^{-n/2} n^{ns-n/2} \frac{\Gamma\left(\frac{1-ns}{2}\right)}{\Gamma\left(\frac{ns-(n-1)}{2}\right)} y^s ds \\ &+ \frac{1}{2\pi i} \int_{(-\sigma_0)} \int_0^\infty f(x) x^{s-1} dx \pi^{-n/2} n^{ns-n/2} \frac{\Gamma\left(\frac{1-ns}{2}\right)}{\Gamma\left(\frac{ns-(n-1)}{2}\right)} H(s) y^s ds. \end{aligned}$$

Using Λ , we shall write the Taylor expansion of $H(\cdot)$ at infinity as

$$H(s) = \sum_{k=1}^K \frac{c'_k}{(s + \Lambda)^k} + O(|s + \Lambda|^{-K-1}).$$

Now, from each term of the sum $\sum_{k=1}^K$ we get an integral

$$\frac{1}{2\pi i} \int_{(-\sigma_0)} \int_0^\infty f(x) x^{s-1} dx \pi^{-n/2} n^{ns-n/2} \frac{\Gamma\left(\frac{1-ns}{2}\right)}{\Gamma\left(\frac{ns-(n-1)}{2}\right)} (s + \Lambda)^{-k} y^s ds.$$

We get a similar integral from the O -term and it will be handled by Lemma 12 below.

We shall consider, for $\nu, k \in \{0, 1, 2, \dots\}$, the more general integral

$$\begin{aligned} \Omega_{\nu,k}(y) = \frac{1}{2\pi i} \int_{(-\sigma_0)} \int_0^\infty f(x) x^{s-1} dx \pi^{-n/2} n^{ns-n/2} \\ \cdot \frac{\Gamma\left(\frac{1-ns}{2}\right)}{\Gamma\left(\frac{ns+1}{2} + \nu - \frac{n}{2}\right)} \cdot (s + \Lambda)^{-k} \cdot y^s ds. \end{aligned}$$

Ultimately, all the main terms will come from J -Bessel functions which occur when $k = 0$, and the error term comes from the case of large $k + \nu$ and the error term in the asymptotics of the J -Bessel functions.

The rest of the argument runs as follows:

1. We first compute asymptotics for $\Omega_{\nu,0}$ via residues. This, in particular, gives the main term which comes from $\Omega_{0,0}$.
2. We then consider the case of large $\nu + k$ by a simple shift of the line of integration and estimates by absolute values. The result will be small. Since the estimates are by absolute values, the argument also works with $(s + \Lambda)^{-k}$ replaced by $O((s + \Lambda)^{-k})$.
3. Finally, we will write $\Omega_{\nu,k}(y)$ in terms of $\Omega_{\nu+1,k-1}(y)$ and $\Omega_{\nu+1,k}(y)$. Iterating this and combining it with the previous two steps gives the desired result by induction on k .

These steps are taken in Lemmas 11, 12 and 13 below. □

Lemma 11. *Let $\nu \in \{0, 1, 2, \dots\}$. Then*

$$\begin{aligned} \Omega_{\nu,0}(y) = -2 \pi^{-n/2} y^{1/2+(1-\nu)/n} n^{-\nu} \\ \cdot \int_0^\infty f(x) x^{(1-\nu)/n-1/2} J_{\nu-n/2}(2n x^{1/n} y^{1/n}) dx. \end{aligned}$$

Proof. The integrand has poles at $1/n, 3/n, 5/n, \dots$, apart possibly those finitely many points where the Γ -factor in the denominator has a pole, but we will not worry about this as formally everything will work out just fine (cf. e.g. Section 5.3 in [25]).

The residue at $s = (2j + 1)/n$, where $j \in \{0, 1, 2, \dots\}$ is

$$\int_0^\infty f(x) x^{(2j+1)/n-1} dx \pi^{-n/2} n^{2j+1-n/2} \cdot \left(-\frac{2}{n}\right) \cdot \frac{(-1)^j}{j!} \cdot \frac{y^{(2j+1)/n}}{\Gamma(j+1+\nu-\frac{n}{2})}.$$

The series of the residues is

$$\begin{aligned} & \sum_{j=0}^\infty \int_0^\infty f(x) x^{(2j+1)/n-1} dx \pi^{-n/2} n^{2j+1-n/2} \cdot \left(-\frac{2}{n}\right) \frac{(-1)^j}{j!} \cdot \frac{y^{(2j+1)/n}}{\Gamma(j+1+\nu-\frac{n}{2})} \\ &= -2 \pi^{-n/2} \int_0^\infty f(x) \sum_{j=0}^\infty \frac{(-1)^j}{j!} \cdot \frac{(2n x^{1/n} y^{1/n})^{2j+\nu-n/2}}{\Gamma(j+1+\nu-\frac{n}{2})} \\ & \quad \cdot 2^{n/2-\nu-2j} \cdot n^{-\nu} x^{2j/n+1/n-1-2j/n-\nu/n+1/2} y^{2j/n+1/n-2j/n-\nu/n+1/2} dx \\ &= -2 \pi^{-n/2} y^{1/2+(1-\nu)/n} n^{-\nu} \int_0^\infty f(x) x^{(1-\nu)/n-1/2} J_{\nu-n/2}(2n x^{1/n} y^{1/n}) dx. \end{aligned}$$

Here the exchange of the summation and integration is allowed because the power series of the J -Bessel function has the usual pleasant properties (cf. Section 5.3 in [25]).

Thus, the result follows by shifting the line of integration in $\Omega_{\nu,0}(y)$ to $\sigma = 2N/n$ for a large positive integer N , and letting $N \rightarrow \infty$. First, the shift of the line of integration by a finite amount is justified by observing that in a vertical strip, the Mellin transform of f is rapidly decaying and by Stirling's formula the quotient of Γ -factors is polynomially bounded, and when moving to right, the Γ -factors give more and more decay.

Thus, it only remains to prove that the integral over the line $\sigma = 2N/n$ tends to zero. The Γ -factor of the numerator can be estimated as follows: We repeatedly use the functional equation $s\Gamma(s) = \Gamma(s+1)$ to shift the argument of the Γ -function to the line $\Re s = 1/2$, and then use the latter part of Theorem 8. This leads to

$$\Gamma\left(\frac{1}{2} - N - \frac{int}{2}\right) \ll \frac{e^{-nt/2}}{(N-1)!}.$$

Similarly the denominator may be estimated by

$$\frac{1}{\Gamma(N + \frac{1}{2} + \nu - \frac{n}{2} + \frac{int}{2})} \ll \frac{e^{nt/2}}{\langle t \rangle^C},$$

where C is a positive real constant. The Mellin transform of f may be estimated by

$$\ll_{L,f} \langle t \rangle^{-L} B^N,$$

where L is an arbitrary fixed positive integer, and B is a large positive real constant which depends on the support of f . Combining these gives the result as the exponential factors cancel each other and the factorial $(N-1)!$ grows faster than anything that remains. \square

We next prove that when $\nu + k$ is large enough, the integrals $\Omega_{\nu,k}(y)$ are rather small.

Lemma 12. *For $\nu + k \geq K + 2$, we have*

$$\Omega_{\nu,k}(y) \ll y^{1/2+1/(2n)} \int_0^\infty |f(x)| x^{1/(2n)-1/2} O((xy)^{-(K+1)/n}) dx,$$

with the implicit constant depending on K , Λ and n . Furthermore, the same result holds, if the factor $(s + \Lambda)^{-k}$ in the definition of $\Omega_{\nu,k}(y)$ is replaced by any function which is analytic and $O((s + \Lambda)^{-k})$ in the closed vertical strip bounded by the lines $\sigma = K/n + 1/(2n) - 1/2$ and $\sigma = -\sigma_0$.

Proof. We shift the line of integration to $\sigma = -X$, where

$$X = K/n + 1/(2n) - 1/2.$$

By Stirling's formula, the integrand is in the t -aspect

$$\ll \frac{\langle t \rangle^{(1+nX)/2}}{\langle t \rangle^{(-nX+1)/2+\nu-n/2}} \cdot \langle t \rangle^{-k} = \langle t \rangle^{nX+n/2-\nu-k}.$$

Therefore, since $nX = K + 1/2 - n/2 \leq -n/2 + \nu + k - 3/2$, we may use Fubini's theorem to obtain

$$\begin{aligned} \Omega_{\nu,k}(y) &= \pi^{-n/2} n^{-n/2} \int_0^\infty f(x) \frac{1}{2\pi i} \int_{(-X)} x^{s-1} n^{ns} \\ &\quad \cdot \frac{\Gamma(\frac{1-ns}{2})}{\Gamma(\frac{ns+1}{2} + \nu - \frac{n}{2})} (s + \Lambda)^{-k} y^s ds dx. \end{aligned}$$

Now the claim follows from estimation by absolute values as the inner integral is

$$\begin{aligned} &\ll x^{-X-1} n^{-nX} y^{-X} \int_{(-X)} \left| \frac{\Gamma(\dots)}{\Gamma(\dots)} \right| \cdot |s + \Lambda|^{-k} ds \ll_X x^{-X-1} y^{-X} \\ &= x^{1/(2n)-1/2-(K+1)/n} y^{1/2+1/(2n)-(K+1)/n}. \end{aligned}$$

□

Lemma 13. *Let $\nu, k \in \{0, 1, 2, \dots\}$, and let K and Λ be as before, and assume that $K \geq \nu + k$. Then*

$$\Omega_{\nu, k}(y) = y^{1/2+1/(2n)} \int_0^{\infty} f(x) x^{1/(2n)-1/2} \mathcal{K}(x, y) dx,$$

where the kernel $\mathcal{K}(x, y)$ has the asymptotics

$$\begin{aligned} \mathcal{K}(x, y) = \sum_{\ell=\nu+k}^K (xy)^{-\ell/n} (c_{\ell}^+ e(\pi^{-1} n x^{1/n} y^{1/n}) + c_{\ell}^- e(-\pi^{-1} n x^{1/n} y^{1/n})) \\ + O((xy)^{-(K+1)/n}). \end{aligned}$$

The constants c_{ℓ}^{\pm} , the kernel \mathcal{K} , and the implicit constant in the O -term depend on n, ν, k, K and Λ .

Proof. We shall perform induction on k . The case $k = 0$ follows immediately from Lemma 11 and the asymptotics for the J -Bessel function. Let us then assume that $k \in \mathbb{Z}_+$ and that the claim has already been proven for smaller values of k .

The key of the induction step lies in writing

$$\frac{1}{s + \Lambda} = \frac{\alpha}{\frac{ns+1}{2} + \nu - \frac{n}{2}} + \frac{\beta}{\left(\frac{ns+1}{2} + \nu - \frac{n}{2}\right)(s + \Lambda)}.$$

Since the right-hand side is

$$= \frac{\alpha s + \alpha \Lambda + \beta}{\left(\frac{ns+1}{2} + \nu - \frac{n}{2}\right)(s + \Lambda)},$$

this is achieved by choosing

$$\alpha = \frac{n}{2} \quad \text{and} \quad \beta = \frac{1}{2} + \nu - \frac{n}{2} - \frac{n\Lambda}{2}.$$

Thus, using the fact that $s \Gamma(s) = \Gamma(s + 1)$, we obtain the recursion formula

$$\Omega_{\nu, k}(y) = \alpha \Omega_{\nu+1, k-1}(y) + \beta \Omega_{\nu+1, k}(y).$$

Applying this formula to the last term and repeating, say, $N \in \mathbb{Z}_+$ times gives

$$\begin{aligned} \Omega_{\nu, k}(y) = \alpha_1 \Omega_{\nu+1, k-1}(y) + \alpha_2 \Omega_{\nu+2, k-1}(y) + \alpha_3 \Omega_{\nu+3, k-1}(y) \\ + \dots + \alpha_N \Omega_{\nu+N, k-1}(y) + \beta_N \Omega_{\nu+N, k}(y). \end{aligned}$$

Here the last term is covered by Lemma 12, provided that $\nu + k + N \geq K + 2$, and all the other terms are covered by the induction assumption. □

6 Proofs of the resonance results

Let us first prove Theorem 1.

Proof of Theorem 1. Write $f(x)$ for $w(x) e(d^{1/n} x/M^{1-1/n})$, and let $\Omega(\cdot)$ be as in Section 5. Theorem 7 says that

$$\sum_{M \leq m \leq M+\Delta} A(m, 1, \dots, 1) w(m) e\left(\frac{d^{1/n} m}{M^{1-1/n}}\right) = \sum_{m=1}^{\infty} \frac{A(1, \dots, 1, m)}{m} \Omega(\pi^n m). \quad (3)$$

The integral $\Omega(\pi^n m)$ will be handled using the asymptotics given in Theorem 10. Let us first treat the error term arising from the kernel \mathcal{K} . For a sufficiently large $K \in \mathbb{Z}_+$, the contribution of the error term to the sum (3) equals

$$\begin{aligned} \ll \sum_{m=1}^{\infty} \frac{|A(1, \dots, 1, m)|}{m^{1/2-1/(2n)}} \int_M^{M+\Delta} |w(x)| x^{1/(2n)-1/2} (xm)^{-(K+1)/n} dx \\ \ll_K \Delta M^{1/(2n)-1/2-(K+1)/n}. \end{aligned}$$

For K fixed and sufficiently large depending on n , this is $\ll 1$. Now we move to the main term coming from the kernel \mathcal{K} . We are left with estimating integrals

$$\int_M^{M+\Delta} g_m^{\pm}(x) w(x) e\left(\frac{d^{1/n} x}{M^{1-1/n}} \pm n x^{1/n} m^{1/n}\right) x^{1/(2n)-1/2} m^{1/2+1/(2n)} dx,$$

where

$$g_m^{\pm}(x) = \sum_{\ell=0}^K c_{\ell}^{\pm} (xm)^{-\ell/n}.$$

The derivative of the phase function, for $d \neq m$ in the case of the minus-sign and for all m otherwise, is

$$\left| \frac{d^{1/n}}{M^{1-1/n}} \pm x^{1/n-1} m^{1/n} \right| \asymp \frac{m^{1/n}}{M^{1-1/n}}.$$

Therefore, using Lemma 9 we get for any $P \in \mathbb{Z}_+$ that

$$\begin{aligned} & \int_M^{M+\Delta} g_m^\pm(x) w(x) e\left(\frac{d^{1/n}x}{M^{1-1/n}} \pm n x^{1/n} m^{1/n}\right) x^{1/(2n)-1/2} m^{1/2+1/(2n)} dx \\ & \ll M^{1/(2n)-1/2} m^{1/(2n)+1/2} \left(\Delta \cdot \frac{m^{1/n}}{M^{1-1/n}}\right)^{-P} \left(1 + \frac{2\Delta}{M}\right)^P \cdot \Delta \\ & \ll_P \Delta^{-P+1} m^{1/(2n)+1/2-P/n} M^{1/(2n)-1/2+((n-1)P)/n}, \end{aligned}$$

for these values of m . The series of these integrals contribute, for a sufficiently large fixed P ,

$$\ll_P \sum_{m=1}^{\infty} \frac{|A(1, \dots, 1, m)|}{m} \cdot \Delta^{-P+1} m^{1/2n+1/2-P/n} M^{1/2n-1/2+(n-1)P/n} \ll_P 1.$$

Substituting this back to the right-hand side of (3) yields

$$\begin{aligned} & \sum_{M \leq m \leq M+\Delta} A(m, 1, \dots, 1) w(m) e\left(\frac{d^{1/n}m}{M^{1-1/n}}\right) = \frac{A(1, \dots, 1, d)}{d^{1/2-1/(2n)}} \\ & \cdot \int_M^{M+\Delta} g_d^-(x) e\left(\frac{d^{1/n}x}{M^{1-1/n}} - n x^{1/n} d^{1/n}\right) w(x) x^{1/(2n)-1/2} dx + O(1). \end{aligned}$$

The term with $\ell = 0$ in the sum $g_d^-(x)$ will give the main term. Next we consider terms with $\ell \geq 1$. Integrating using absolute values gives

$$\begin{aligned} & \int_M^{M+\Delta} c_\ell^- w(x) e\left(\frac{d^{1/n}x}{M^{1-1/n}} - n x^{1/n} d^{1/n}\right) x^{-\ell/n} x^{1/(2n)-1/2} dx \\ & \ll \Delta M^{-\ell/n+1/(2n)-1/2} \ll \Delta M^{-1/2-1/(2n)}. \end{aligned}$$

Hence

$$\begin{aligned} & \sum_{M \leq m \leq M+\Delta} A(m, 1, \dots, 1) w(m) e\left(\frac{d^{1/n}m}{M^{1-1/n}}\right) \\ & = \frac{A(1, \dots, 1, d)}{d^{1/2-1/(2n)}} c_0^- \int_M^{M+\Delta} w(x) e\left(\frac{d^{1/n}x}{M^{1-1/n}} - n x^{1/n} d^{1/n}\right) x^{1/(2n)-1/2} dx \\ & \quad + O(\Delta M^{-1/2-1/(2n)}) + O(1) \end{aligned}$$

$$\begin{aligned}
&= \frac{A(1, \dots, 1, d)}{d^{1/2-1/(2n)} \sqrt{n}} e\left(-\frac{n+3}{8}\right) \\
&\quad \cdot \int_M^{M+\Delta} w(x) e\left(\frac{d^{1/n} x}{M^{1-1/n}} - n x^{1/n} d^{1/n}\right) x^{1/(2n)-1/2} dx \\
&\quad + O(\Delta M^{-1/2-1/(2n)}),
\end{aligned}$$

which is what we wanted. \square

Proof of Theorem 4. Write $f(x)$ for $w(x) e(n d^{1/n} x^{1/n})$, and let $\Omega(\cdot)$ be as in Section 5. Theorem 7 says that

$$\sum_{M \leq m \leq M+\Delta} A(m, 1, \dots, 1) w(m) e(n d^{1/n} m^{1/n}) = \sum_{m=1}^{\infty} \frac{A(1, \dots, 1, m)}{m} \Omega(\pi^n m). \quad (4)$$

In this case, the precise expression of Ω looks like the following:

$$\Omega(\pi^n m) = y^{1/2+1/(2n)} \int_M^{M+\Delta} w(x) e(n d^{1/n} x^{1/n}) x^{1/(2n)-1/2} \mathcal{K}(x, \pi^n m) dx.$$

The integral $\Omega(\pi^n m)$ will be handled using the asymptotics given in Theorem 10. The error term arising from the kernel \mathcal{K} is treated precisely the same way (since we are only interested in the absolute values, it even yields the same expression) as in the proof of Theorem 1. Now we move to the main term coming from the kernel \mathcal{K} . We are left with estimating integrals

$$\int_M^{M+\Delta} g_m^\pm(x) w(x) e(n d^{1/n} x^{1/n} \pm n x^{1/n} m^{1/n}) x^{1/(2n)-1/2} m^{1/2+1/(2n)} dx,$$

where

$$g_m^\pm(x) = \sum_{\ell=0}^K c_\ell^\pm (xm)^{-\ell/n}.$$

The derivative of the phase function, for $d \neq m$ in the case of the minus-sign and for all m otherwise, is

$$|x^{1/n-1} d^{1/n} \pm x^{1/n-1} m^{1/n}| = \left| \frac{d^{1/n} \pm m^{1/n}}{x^{1-1/n}} \right| \asymp \frac{m^{1/n}}{M^{1-1/n}},$$

since d is fixed. Therefore, just like in the proof of Theorem 1, using Lemma 9 we get for any $P \in \mathbb{Z}_+$ that

$$\begin{aligned} & \int_M^{M+\Delta} g_m^\pm(x) w(x) e(n d^{1/n} x^{1/n} \pm n x^{1/n} m^{1/n}) x^{1/(2n)-1/2} m^{1/2+1/(2n)} dx \\ & \ll M^{1/(2n)-1/2} m^{1/(2n)+1/2} \left(\Delta \cdot \frac{m^{1/n}}{M^{1-1/n}} \right)^{-P} \left(1 + \frac{2\Delta}{M} \right)^P \cdot \Delta \\ & \ll_P \Delta^{-P+1} m^{1/(2n)+1/2-P/n} M^{1/(2n)-1/2+((n-1)P)/n}, \end{aligned}$$

for these values of m . The series of these integrals contribute, for a sufficiently large fixed P ,

$$\ll_P \sum_{m=1}^{\infty} \frac{|A(1, \dots, 1, m)|}{m} \cdot \Delta^{-P+1} m^{1/2n+1/2-P/n} M^{1/2n-1/2+(n-1)P/n} \ll_P 1.$$

Substituting this back to the right-hand side of (4) yields

$$\begin{aligned} & \sum_{M \leq m \leq M+\Delta} A(m, 1, \dots, 1) w(m) e(n d^{1/n} m^{1/n}) = \frac{A(1, \dots, 1, d)}{d^{1/2-1/(2n)}} \\ & \cdot \int_M^{M+\Delta} g_d^-(x) e(n d^{1/n} x^{1/n} - n x^{1/n} d^{1/n}) w(x) x^{1/(2n)-1/2} dx + O(1). \end{aligned}$$

Notice that since $d^{1/n} x^{1/n} - x^{1/n} d^{1/n} = 0$, the integrand does not oscillate at all, which is the reason why Δ can be taken up to M , instead of the more strict constraint in Theorem 1, and simplifies to the form

$$\int_M^{M+\Delta} g_d^-(x) w(x) x^{1/(2n)-1/2} dx$$

The term with $\ell = 0$ in the sum $g_d^-(x)$ will give the main term. Next we consider terms with $\ell \geq 1$. Integrating using absolute values gives

$$\int_M^{M+\Delta} c_\ell^- w(x) x^{-\ell/n} x^{1/(2n)-1/2} dx \ll \Delta M^{-\ell/n+1/(2n)-1/2} \ll \Delta M^{-1/2-1/(2n)}.$$

Hence

$$\begin{aligned}
& \sum_{M \leq m \leq M+\Delta} A(m, 1, \dots, 1) w(m) e(n d^{1/n} m^{1/n}) \\
&= \frac{A(1, \dots, 1, d)}{d^{1/2-1/(2n)}} c_0^- \int_M^{M+\Delta} w(x) x^{1/(2n)-1/2} dx + O(\Delta M^{-1/2-1/(2n)} + 1) \\
&= \frac{A(1, \dots, 1, d)}{d^{1/2-1/(2n)} \sqrt{n}} e\left(-\frac{n+3}{8}\right) \int_M^{M+\Delta} w(x) x^{1/(2n)-1/2} dx + O(\Delta M^{-1/2-1/(2n)}),
\end{aligned}$$

which is what we wanted. \square

7 Ω -results

From Corollary 2, we also get large values for unweighted short sums as follows.

Corollary 14. *Let $d \in \mathbb{Z}_+$ be fixed and such that $A(1, \dots, 1, d) \neq 0$. Then there exists $\Delta \asymp M^{1-1/(2n)}$ such that*

$$\sum_{M \leq m \leq M+\Delta} A(m, 1, \dots, 1) e\left(\frac{d^{1/n}}{M^{1-1/n}} m\right) \gg M^{1/2}.$$

Proof. Let $w(x)$ be as in Corollary 2, and assume

$$\sum_{M \leq m \leq M+\Delta} A(m, 1, \dots, 1) e\left(\frac{d^{1/n}}{M^{(n-1)/n}} m\right) = o(M^{1/2})$$

for all $\Delta \asymp M^{1-1/(2n)}$ and that $w(x) = 1$ when $x \in [M_1, M_2]$. Further assume $M_1 - M \asymp M_2 - M_1 \asymp M + \Delta - M_2$. Now

$$\begin{aligned}
& \sum_{M \leq m \leq M+\Delta} A(m, 1, \dots, 1) e\left(\frac{d^{1/n}}{M^{1-1/n}} m\right) w(m) \\
&= \left(\sum_{M \leq m \leq M_1} + \sum_{M_2 \leq m \leq M+\Delta} \right) A(m, 1, \dots, 1) e\left(\frac{d^{1/n}}{M^{1-1/n}} m\right) w(m) \\
&\quad + \sum_{M_1 < m < M_2} A(m, 1, \dots, 1) e\left(\frac{d^{1/n}}{M^{1-1/n}} m\right) \\
&= \left(\sum_{M \leq m \leq M_1} + \sum_{M_2 \leq m \leq M+\Delta} \right) A(m, 1, \dots, 1) e\left(\frac{d^{1/n}}{M^{1-1/n}} m\right) w(m) \\
&\quad + o(M^{1/2}).
\end{aligned}$$

We may now use partial summation to bound the sums. We obtain

$$\begin{aligned}
& \left(\sum_{M \leq m \leq M_1} + \sum_{M_2 \leq m \leq M+\Delta} \right) A(m, 1, \dots, 1) e\left(\frac{d^{1/n}}{M^{1-1/n}} m\right) w(m) \\
&= w(M_1) \sum_{M \leq m \leq M_1} A(m, 1, \dots, 1) e\left(\frac{d^{1/n}}{M^{1-1/n}} m\right) \\
&\quad + w(M+\Delta) \sum_{M_2 \leq m \leq M+\Delta} A(m, 1, \dots, 1) e\left(\frac{d^{1/n}}{M^{1-1/n}} m\right) \\
&\quad - \int_M^{M_1} \left(\sum_{M \leq m \leq t} A(m, 1, \dots, 1) e\left(\frac{d^{1/n}}{M^{1-1/n}} m\right) \right) w'(t) dt \\
&\quad - \int_{M_2}^{M+\Delta} \left(\sum_{M_2 \leq m \leq t} A(m, 1, \dots, 1) e\left(\frac{d^{1/n}}{M^{1-1/n}} m\right) \right) w'(t) dt \\
&= o(M^{1/2}) + o\left(\int_M^{M_1} M^{1/2} \Delta^{-1} dt\right) + o\left(\int_{M_2}^{M+\Delta} M^{1/2} \Delta^{-1} dt\right) = o(M^{1/2}),
\end{aligned}$$

which leads to a contradiction with Corollary 2. This proves the corollary. \square

Since a sum can be estimated by splitting it into shorter sums and then adding up upper bounds for the subsums, we get the Ω -result also for shorter sums:

Corollary 15. *Let $d \in \mathbb{Z}_+$ be fixed and such that $A(1, \dots, 1, d) \neq 0$. Then, for any $\gamma \in [1/2 - 1/(2n), 1 - 1/(2n)]$, there exists $\Delta \asymp M^\gamma$ such that*

$$\sum_{M \leq m \leq M+\Delta} A(m, 1, \dots, 1) e\left(\frac{d^{1/n}}{M^{1-1/n}} m\right) \gg \Delta M^{1/(2n)-1/2}.$$

Let us state a lemma about the size of exponential sums before proving Theorem 5.

Lemma 16. *We have*

$$\sum_{0 \leq h \leq \Delta-1} e\left(\frac{d^{1/n}}{M^{1-1/n}} h\right) \asymp \Delta$$

when $\Delta = o(M^{(n-1)/n})$ and $d \in \mathbb{Z}_+$ is fixed.

Proof. Since $\sum_{0 \leq h \leq \Delta-1} e\left(\frac{d^{1/n}}{M^{1-1/n}} h\right)$ is a geometric sum, we have

$$\sum_{0 \leq h \leq \Delta-1} e\left(\frac{d^{1/n}}{M^{1-1/n}} h\right) = \frac{e\left(\frac{d^{1/n}}{M^{(n-1)/n}} \lfloor \Delta \rfloor\right) - 1}{e\left(\frac{d^{1/n}}{M^{(n-1)/n}}\right) - 1} \asymp \frac{d^{1/n} M^{-(n-1)/n} \Delta}{d^{1/n} M^{-(n-1)/n}} = \Delta.$$

□

We are now ready to move to the proof of Theorem 5.

Proof of Theorem 5. Let $U \asymp M^{1-1/(2n)}$ be such that

$$\sum_{M \leq m \leq M+U} A(m, 1, \dots, 1) e\left(\frac{d^{1/n}}{M^{(n-1)/n}} m\right) \gg M^{1/2}.$$

Without loss of generality, we may assume that M , U and Δ are integers. Now

$$\left(\sum_{0 \leq h \leq \Delta-1} e\left(\frac{d^{1/n}}{M^{1-1/n}} h\right)\right) \left(\sum_{M \leq m \leq M+U} A(m, 1, \dots, 1) e\left(\frac{d^{1/n}}{M^{(n-1)/n}} m\right)\right) \gg M^{1/2} \Delta.$$

On the other hand, we have

$$\begin{aligned} & \left(\sum_{0 \leq h \leq \Delta-1} e\left(\frac{d^{1/n}}{M^{1-1/n}} h\right)\right) \left(\sum_{M \leq m \leq M+U} A(m, 1, \dots, 1) e\left(\frac{d^{1/n}}{M^{(n-1)/n}} m\right)\right) \\ &= \sum_{\substack{0 \leq h \leq \Delta-1 \\ M \leq m \leq M+U}} A(m, 1, \dots, 1) e\left(\frac{d^{1/n}}{M^{1-1/n}} (m+h)\right) \\ &= e\left(\frac{d^{1/n}}{M^{1-1/n}} M\right) A(M, 1, \dots, 1) \\ & \quad + e\left(\frac{d^{1/n}}{M^{1-1/n}} (M+1)\right) (A(M, 1, \dots, 1) + A(M+1, 1, \dots, 1)) + \dots \\ & \quad + e\left(\frac{d^{1/n}}{M^{1-1/n}} (M+\Delta-1)\right) \sum_{M \leq m \leq M+\Delta-1} A(m, 1, \dots, 1) \\ & \quad \dots \\ & \quad + e\left(\frac{d^{1/n}}{M^{1-1/n}} (M+U)\right) \sum_{-\Delta+1+M+U \leq m \leq M+U} A(m, 1, \dots, 1) \\ & \quad + \dots + e\left(\frac{d^{1/n}}{M^{1-1/n}} (M+U+\Delta-1)\right) A(M+U+\Delta-1, 1, \dots, 1). \end{aligned}$$

The sums that are of length $< \Delta$ are at most of size $\Delta M^{\vartheta+\varepsilon}$. Hence, their total contribution is $\ll \Delta^2 M^{\vartheta+\varepsilon}$, so that when $\Delta^2 M^{\vartheta+\varepsilon} = o(M^{1/2} \Delta)$, i.e. when $\Delta = o(M^{1/2-\vartheta-\varepsilon})$, we have

$$e\left(\frac{d^{1/n}}{M^{1-1/n}}(M + \Delta - 1)\right) \sum_{M \leq m \leq M+\Delta-1} A(m, 1, \dots, 1) + \dots \\ + e\left(\frac{d^{1/n}}{M^{1-1/n}}(M + U)\right) \sum_{-\Delta+1+M+U \leq m \leq M+U} A(m, 1, \dots, 1) \gg M^{1/2} \Delta.$$

Since the number of sums is U , there must be a sum of size

$$\gg M^{1/2} \Delta U^{-1} \asymp M^{1/2} \Delta M^{1/(2n)-1} = \Delta M^{1/(2n)-1/2}.$$

□

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