On Non-Scattering Energies

Esa V. Vesalainen

Doctoral dissertation

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Department of Mathematics and Statistics
Faculty of Science
University of Helsinki

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Advisors

Prof. Mikko Salo
Department of Mathematics and Statistics
University of Jyväskylä

Prof. Lassi Päivärinta
Department of Mathematics and Statistics
University of Helsinki

Pre-examiners

Prof. Fioralba Cakoni
Department of Mathematical Sciences
University of Delaware

Prof. Valery Serov
Department of Mathematical Sciences
University of Oulu

Opponent

Prof. John Sylvester
Department of Mathematics
University of Washington

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Contents

Abstract

In this thesis, we extend the theory of non-scattering energies on two fronts. First, we shall consider the discreteness of non-scattering energies corresponding to non-compactly supported potentials using the approach via transmission eigenvalues and fourth-order operators. The method requires the support of the potential to exhibit certain compact Sobolev embedding and to be contained in a half-space and the potential to have controlled polynomial or exponential decay at infinity. Also, in order to connect the non-scattering energies to the fourth-order operators, a generalization of the classical Rellich theorem to unbounded domains is required. This is of independent interest, and we obtain several such results, including a discrete analogue.

Our second contribution (joint work with L. Päivärinta and M. Salo) is extending a recent result on non-existence of non-scattering energies for potentials with rectangular corners to arbitrary corners of angle smaller than π in two dimensions, and to prove in three dimensions that the set of strictly convex circular conical corners for which non-scattering energies might exist is at most countable.

This thesis consists of the papers

I. Vesalainen, E. V.: *Transmission eigenvalues for non-compactly supported potentials*, Inverse Problems, 29 (2013), 104006, 1–11.

II. Vesalainen, E. V.: *Rellich type theorems for unbounded domains*, to appear in Inverse Problems and Imaging. Preprint available at arXiv:1401.4531 [math.AP].

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The author of this thesis had an equal role in the research and writing of the joint article.

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Introduction

1 Scattering and non-scattering energies

1.1 Scattering theory

Our objects of study arise from scattering theory. More precisely, time independent scattering theory for short-range potentials, which models e.g. two-body quantum scattering, acoustic scattering, and some classical electromagnetic scattering situations (for a general reference, see e.g. [10]). Here one is concerned with the situation where, at a fixed energy or wavenumber $\lambda \in \mathbb{R}_+$, an incoming wave w, which is a solution to the free equation

$$(-\Delta - \lambda)\, w = 0,$$

is scattered by some perturbation of the flat homogeneous background. Here this perturbation will be modeled by a real-valued function V in $\mathbb{R}^n$ having enough decay at infinity. The total wave v, which models the "actual" wave, then solves the perturbed equation

$$(-\Delta + V - \lambda)\, v = 0.$$

In acoustic and electromagnetic scattering, one has λV instead of V. Of course, the two waves v and w must be linked together and the connection is given by the Sommerfeld radiation condition. The upshot will be that the difference u of v and w, the so-called scattered wave, will have an asymptotic expansion of the shape

$$u(x) = v(x) - w(x) = A\left(\frac{x}{|x|}\right) \frac{e^{i\sqrt{\lambda}|x|}}{|x|^{(n-1)/2}} + \text{error},$$

where A depends on λ and w, and where the error term decays more rapidly than the main term. The point here is that in the main term the dependences on the radial and angular variables are neatly separated, and in practical applications one usually measures the scattering amplitude or far-field pattern A, or its absolute value $|A|$.

1.2 Non-scattering energies

It is a natural question whether we can have $A \equiv 0$ for some $w \not\equiv 0$? This would mean that the main term of the scattered wave vanishes at infinity, meaning that the perturbation, for the special incident wave in question, is not seen far away. Values of $\lambda \in \mathbb{R}_+$ for which such an incident wave w exist, are called non-scattering energies (or appropriately, wavenumbers) of V.

Of course, the functions u, v and w will be from some specific function spaces. To be precise, $\lambda \in \mathbb{R}_+$ is a non-scattering energy for a short-range potential V if there exist non-zero functions $v, w \in B_2^*$ solving the equations

$$\begin{cases} (-\Delta + V - \lambda)\, v = 0, \\ (-\Delta - \lambda)\, w = 0, \end{cases}$$

in $\mathbb{R}^n$, and having the same asymptotic behaviour at infinity in the sense that $u = v - w \in \mathring{B}_2^*$. Here

$$B_2^* = \left\{ u \in B^* \;\middle|\; \partial^\alpha u \in B^* \text{ for multi-indices } \alpha \text{ with } |\alpha| \leqslant 2 \right\},$$

and similarly

$$\mathring{B}_2^* = \left\{ u \in \mathring{B}_2^* \;\middle|\; \partial^\alpha u \in \mathring{B}_2^* \text{ for multi-indices } \alpha \text{ with } |\alpha| \leqslant 2 \right\},$$

where B^* and $\mathring{B}^*$ are the Agmon–Hörmander spaces

$$B^* = \left\{ u \in L^2_{\mathrm{loc}}(\mathbb{R}^n) \;\middle|\; \sup_{R>1} \frac{1}{R} \int\limits_{B(0,R)} |u|^2 < \infty \right\}$$

and

$$\mathring{B}^* = \left\{ u \in B^* \;\middle|\; \lim_{R\longrightarrow\infty} \frac{1}{R} \int\limits_{B(0,R)} |u|^2 = 0 \right\}.$$

A function $V \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ is a short-range potential for instance when $V(\cdot) \ll \langle\cdot\rangle^{-\alpha}$ in $\mathbb{R}^n$ for some $\alpha \in\,]1, \infty[$. For a presentation of short-range scattering theory in terms of the Agmon–Hörmander spaces, see e.g. Chapter XIV of [18] and the first sections of [30]. Also, the articles [3] and [4] are recommended.

Results on the existence of non-scattering energies are scarce. Essentially only two general results are known: For compactly supported radial potentials the set of non-scattering energies is an infinite discrete set accumulating at infinity [11], and for compactly supported potentials with rectangular corners, Blåsten, Päivärinta and Sylvester recently proved that the set of non-scattering energies is empty [5]. In addition to these, the many results on the discreteness of transmission eigenvalues for various compactly supported potentials also imply the discreteness of non-scattering energies for some of the corresponding potentials. It is not yet known if non-scattering energies can exist for non-radial potentials.

We would like to mention the related topic of transparent potentials: there one considers (at a fixed energy) potentials for which A vanishes for all w. The knowledge of transparent potentials is more extensive. In particular, several constructions of such radial potentials have been given, see e.g. the works of Regge [31], Newton [28], Sabatier [35], Grinevich and Manakov [13], and Grinevich and Novikov [14].

2 Discreteness via fourth-order operators

2.1 The compactly supported story

Discreteness of the set of non-scattering energies tends to be a much more attainable goal than knowledge of existence or non-existence. The first key step towards that goal (for compactly supported V) is supplied by Rellich's classical uniqueness theorem which is the following:

Theorem 1. *Let $u \in \mathring{B}_2^*$ solve the equation $(-\Delta - \lambda)\, u = f$, where $\lambda \in \mathbb{R}_+$ and $f \in L^2(\mathbb{R}^n)$ is compactly supported. Then u also is compactly supported.*

This was first proved (though with a bit different decay condition) independently by Rellich [32] and Vekua [43] in 1943. Of the succeeding work, which includes generalizations of this result to more general constant coefficient differential operators, we would like to mention the work of Trèves [41], Littman [24, 25, 26], Murata [27] and Hörmander [17]. Section 8 of [16] is also interesting.

Now, assume that V is compactly supported. The equations for v and w imply that the scattered wave u solves the equation

$$(-\Delta - \lambda)\, u = -Vv.$$

If furthermore $A \equiv 0$, then u will satisfy the decay condition in Theorem 1, and so $u = v - w$ will vanish outside a compact set. If the support of V is essentially contained in some suitable open domain Ω, the unique continuation principle for the free Helmholtz equation allows us to conclude that actually

$$\begin{cases} (-\Delta + V - \lambda)\, v = 0 & \text{in } \Omega, \\ (-\Delta - \lambda)\, w = 0 & \text{in } \Omega, \\ v - w \in H_0^2(\Omega). \end{cases}$$

This system, called the interior transmission problem, is a non-self-adjoint eigenvalue problem for λ, and the values of λ, for which this system has non-trivial L^2-solutions, are called (interior) transmission eigenvalues.

The non-scattering energies and transmission eigenvalues first appeared in the papers of Colton and Monk [11] and Kirsch [22]. In [9] Colton, Kirsch and Päivärinta proved the discreteness of transmission eigenvalues (and non-scattering energies) for potentials that may even be mildly degenerate. The early papers on the topic also considered, among other things, radial potentials; for more on this, we refer to the article of Colton, Päivärinta and Sylvester [12].

In recent years, there has been a surge of interest in the topic starting with the general existence results of Päivärinta and Sylvester [30], who established existence of transmission eigenvalues for a large class of potentials, and Cakoni, Gintides and Haddar [6], who established for acoustic scattering, that actually the set of transmission eigenvalues must be infinite.

For potentials more general than the radial ones, a very common approach to proving discreteness and other properties has been via quadratic forms: the scattered wave solves the fourth-order equation

$$(-\Delta + V - \lambda)\, \frac{1}{V}\, (-\Delta - \lambda)\, u = 0,$$

and this can be handled nicely with quadratic forms (or with variational formulations) and analytic perturbation theory. We shall discuss this in more detail below.

Recently, other more general approaches, not involving the fourth-order equation, to proving discreteness and many other results have been introduced by Sylvester [39], Robbiano [33], and Lakshtanov and Vainberg [23].

For more information on transmission eigenvalues, we recommend the survey of Cakoni and Haddar [7] and their editorial [8] as well as the articles mentioned there and their references.

2.2 Non-compactly supported potentials

Most of the work on non-scattering energies and transmission eigenvalues deals with compactly supported potentials V. However, the basic short-range scattering theory only requires V to have enough decay at infinity, essentially something like $V(x) \ll \langle x \rangle^{-1-\varepsilon}$. Thus, it makes perfect sense to study non-scattering energies for non-compactly supported potentials.

In [44, 45] we take first steps into the direction of non-compact supports by considering non-scattering energies and transmission eigenvalues for non-compact Ω which are nearly compact in the sense that they have a suitable compact Sobolev embedding, and for potentials V taking only positive real values and having a certain kind of controlled asymptotic behaviour. For these potentials, we prove the basic discreteness result using the approach via fourth-order operators described above. The more usual case of bounded Ω with a positive real-valued potential, which is bounded and bounded away from zero, is covered as a special case. The potential V may decay polynomially or exponentially fast at infinity. The latter case is simpler, and in the following we shall focus on it. The precise statement of the result is the following.

Theorem 2. *Let $V \in L^\infty(\mathbb{R}^n)$ take only nonnegative real values, and let $\Omega \subseteq \mathbb{R}^{n-1} \times \mathbb{R}_+$ be a non-empty open set for which the Sobolev embedding $H_0^2(\Omega) \longrightarrow L^2(\Omega)$ is compact. Assume the following:*

I. *$V(\cdot) \asymp e^{-\gamma_0 \langle \cdot \rangle}$ in Ω for some $\gamma_0 \in \mathbb{R}_+$ with $\gamma_0 \ll_n 1$, and V vanishes in $\mathbb{R}^n \setminus \Omega$.*

II. *The complement of Ω in $\mathbb{R}^n$ has a connected interior and is the closure of the interior.*

Then the set of non-scattering energies for V is a discrete subset of $[0, \infty[$ and each of them is of finite multiplicity.

Here the purpose of the condition II is the following: we first prove that the scattered wave u corresponding to a hypothetical non-scattering energy must vanish in the lower half-space. Then the condition II allows us to use the unique continuation principle for the Helmholtz equation to conclude that u vanishes in $\mathbb{R}^n \setminus \Omega$.

The condition on the compact embedding $H_0^2(\Omega) \longrightarrow L^2(\Omega)$ is satisfied for $n \leqslant 3$ if and only if the domain Ω does not contain an infinite sequence of pairwise disjoint congruence balls (see e.g. [1] or Chapter 6 in [2]). For $n \geqslant 4$ the situation is more complicated.

2.3 Rellich type theorems for unbounded domains

Already for the first step, that of reducing the equations which hold in $\mathbb{R}^n$ to equations in the support of the potential, the Rellich type theorem, Theorem 1, must be generalized. In [45] we give two results of this kind: the first is for exponentially decaying inhomogeneities, the second is for polynomially decaying potentials but for domains that are not only contained in a half-space but also grow exponentially thin at infinity. These results are proved with a more traditional complex variables argument [41, 24, 25, 26, 17].

We sketch the proof for the case of exponential decay. The result is as follows.

Theorem 3. *Let $u \in \mathring{B}_2^*$ solve*

$$(-\Delta - \lambda)\, u = f,$$

where $\lambda \in \mathbb{R}_+$ and $f \in e^{-\gamma_0 \langle \cdot \rangle} L^2(\mathbb{R}^n)$ for some $\gamma_0 \in \mathbb{R}_+$, and suppose that f vanishes in $\mathbb{R}^{n-1} \times \mathbb{R}_-$. Then also u vanishes in $\mathbb{R}^{n-1} \times \mathbb{R}_-$.

The fundamental idea is to take Fourier transforms, leading to

$$(p - \lambda)\, \widehat{u} = \widehat{f},$$

where $p(\xi) = 4\pi^2 \xi^2$, which holds in $\mathbb{R}^n$. From basic scattering theory, we know that $\widehat{f}$ vanishes on the real sphere

$$M_\lambda^{\mathbb{R}} = \big\{ \xi \in \mathbb{R}^n \;\big|\; p(\xi) = \lambda \big\}.$$

Also, the Fourier transform $\widehat{f}$ extends to an analytic function in

$$D = \big\{ \zeta \in \mathbb{C}^n \;\big|\; |\Im \zeta| < \gamma_0 \big\}.$$

From this, it follows, by flattening the spheres $M_\lambda^{\mathbb{R}}$ and $M_\lambda^{\mathbb{C}}$ locally, that $\widehat{f}$ vanishes on the intersection $D \cap M_\lambda^{\mathbb{C}}$, where $M_\lambda^{\mathbb{C}}$ is the complex sphere

$$M_\lambda^{\mathbb{C}} = \big\{ \zeta \in \mathbb{C}^n \;\big|\; p(\zeta) = \lambda \big\},$$

and furthermore, $\widehat{f}/(p - \lambda)$ extends to an analytic function in D.

Next, fix a point $\xi' \in \mathbb{R}^{n-1}$ with $|\xi'| < \sqrt{\lambda}/2\pi$, and write for simplicity $q(\cdot)$ for $p(\xi', \cdot)$. Now the expression $q(\cdot) - \lambda$ has only two simple zeros $\pm\mu$.

Since f vanishes in $\mathbb{R}^{n-1} \times \mathbb{R}_-$, the Fourier transform $F' f(\xi', \cdot)$ vanishes in $\mathbb{R}_-$, so that the classical half-line Paley–Wiener theorem (see e.g. Thm. 19.2 in

[34]), $\widehat{f}$ has an analytic extension in the last variable to $\mathbb{R} \times i\,]-\infty, \gamma_0[$, and

$$\int_{-\infty}^{\infty} \left|\widehat{f}(\xi', \xi_n - i\eta)\right|^2 \mathrm{d}\xi_n \ll \int_{-\infty}^{\infty} \left|\widehat{f}(\xi', \xi_n)\right|^2 \mathrm{d}\xi_n < \infty$$

for $\eta \in \mathbb{R}_-$. Since $1/(q-\lambda)$ vanishes at infinity uniformly, and since the zeros of $F' f(\xi', \cdot)$ at $\pm\mu$ cancel the simple poles of $1/(q-\lambda)$, also

$$\int_{-\infty}^{\infty} \left|\widehat{u}(\xi', \xi_n - i\eta)\right|^2 \mathrm{d}\xi_n$$

is uniformly bounded by some constant not depending on $\eta \in \mathbb{R}_-$.

Now the one-dimensional half-line Paley–Wiener theorem says that

$$F' u(\xi', x_n) = F_n^{-1} \widehat{u}(\xi', x_n) = 0$$

for all ξ' near the origin and all $x_n \in \mathbb{R}_-$. Since $F'u$ is analytic with respect of the first $n-1$ variables, $F'u(\xi', x_n) = 0$ for all $\xi' \in \mathbb{R}^{n-1}$ and all $x_n \in \mathbb{R}_-$. Finally, taking inverse Fourier transform gives the conclusion that u vanishes in $\mathbb{R}^{n-1} \times \mathbb{R}_-$.

2.4 Some spin-offs of generalizing the Rellich type theorem

Studying the problem of generalizing the Rellich theorem lead to two other interesting things not directly relevant to non-scattering energies. First, we proved a generalized Rellich type theorem where instead of a compactly supported inhomogeneity f, we consider f that is superexponentially decaying and vanishes in a half-space. The conclusion will then again be that the solution u also vanishes in the same half-space. The interesting novelty is that we give a new kind of proof for this result based on real variable techniques, first deriving a Carleman estimate weighted exponentially in one direction from an estimate of Sylvester and Uhlmann [40] and then arguing immediately from it.

As a second aside, and to provide a point of comparison, we present a generalization of the discrete Rellich type theorem of Isozaki and Morioka. A theorem analogous to Theorem 1 also exists for the discrete Laplacian (defined precisely in the paper [45]):

Theorem 4. *Let $u\colon \mathbb{Z}^n \longrightarrow \mathbb{C}$ be a solution to $(-\Delta_{\mathrm{disc}} - \lambda)\, u = f$, where*

$$\frac{1}{R} \sum_{\xi \in \mathbb{Z}^n, |\xi| \leqslant R} |u(\xi)|^2 \longrightarrow 0,$$

as $R \longrightarrow \infty$, and $f \in \ell^2(\mathbb{Z}^n)$ is non-zero only at finitely many points of $\mathbb{Z}^n$, and $\lambda \in]0, n[$. Then u also is non-zero only at finitely many points of $\mathbb{Z}^n$.

This theorem was proved recently by Isozaki and Morioka [19]. A less general version of the result was implicit in the work of Shaban and Vainberg [38].

It turns out that for superexponentially decaying potentials, one gets a much stronger result than in the continuous case: we not only can consider vanishing in half-spaces but vanishing in suitable cones. The proof depends heavily on the arguments in [19] which are first used to show that the solution must be superexponentially decaying. After this, the Rellich type conclusion follows from a repeated application of the definition of the discrete Laplacian.

2.5 Fourth-order operators for non-compactly supported potentials

We can now describe the structure of the proof of Theorem 2. So, assume that $\lambda \in \mathbb{R}_+$ is a non-scattering energy corresponding to total and incident waves $v, w \in B_2^*$, and scattered wave $u = v - w \in \mathring{B}_2^*$. Since

$$(-\Delta - \lambda)\, u = -Vv,$$

in $\mathbb{R}^n$, Theorem 3 guarantees that u vanishes in the lower half-space $\mathbb{R}^{n-1} \times \mathbb{R}_-$ and the condition II of Theorem 2 and the unique continuation principle for the Helmholtz equation guarantees that u vanishes in $\mathbb{R}^n \setminus \Omega$. We therefore end up with the system

$$\begin{cases} (-\Delta + V - \lambda)\, v = 0, \\ (-\Delta - \lambda)\, w = 0, \end{cases}$$

which now holds in Ω. Since V is locally bounded away from zero in Ω, we get for u the fourth-order equation

$$(-\Delta + V - \lambda)\, \frac{1}{V}\, (-\Delta - \lambda)\, u = 0,$$

which again holds in Ω. It turns out that u belongs to a Sobolev space weighted essentially by $V^{-1/2}$. I.e. we have $u \in H_V$, where

$$H_V = \big\{ u \in L_V \;\big|\; \partial^\alpha u \in L_V \text{ for } |\alpha| \leqslant 2 \big\},$$

where in turn

$$L_V = \big\{ u \in L^2_{\text{loc}}(\Omega) \;\big|\; V^{-1/2}\, u \in L^2(\Omega) \big\}.$$

We equip L_V and H_V with the obvious weighted norms.

We shall modify the situation further: the existence of a non-trivial solution $u \in H_V$ to the fourth-order equation is equivalent to the existence of a non-trivial solution $u \in H_V$ to $Q_\lambda(u) = 0$, where Q_λ is the quadratic form

$$Q_\lambda(u) = \int_\Omega \left(-\Delta + V - \overline{\lambda}\right) \overline{u} \cdot \frac{1}{V} \left(-\Delta - \lambda\right) u.$$

This is well-defined for all $\lambda \in \mathbb{C}$ and $u \in H_V$. We let the domain of Q_λ be H_V, and we consider Q_λ as a quadratic form of the Hilbert space L_V.

Our manner of using quadratic forms to establish discreteness is a close relative of the application of quadratic forms to degenerate and singular potentials in the works of Colton, Kirsch and Päivärinta [9], Serov and Sylvester[37], Serov [36], and Hickmann [15].

Now, the discreteness will be obtained just by studying the very basic properties of Q_λ. It is not too hard to verify that Q_λ is in fact something called an entire self-adjoint analytic family of quadratic forms of type (a) with compact resolvent. We recommend Kato's presentation [20] for the related basic theory. The key lemma is the weighted estimate

$$\left\|u\right\|_{H_V} \asymp_\lambda \left\|(-\Delta - \lambda)\, u\right\|_{L_V} + \left\|u\right\|_{L_V},$$

true for all $u \in H_V$ and any $\lambda \in \mathbb{R}_+$, which is established using arguments from Appendix A of [3].

By the theory of quadratic forms and analytic perturbation theory, each Q_λ corresponds to a unique closed operator T_λ of L_V and we immediately get certain excellent properties for T_λ. In particular, for $\lambda \in \mathbb{R}$, the quadratic form $Q_\lambda(u)$ corresponds to a unique self-adjoint operator T_λ of the Hilbert space L_V with compact resolvent, and the above fourth-order equation has a non-trivial H_V-solution if and only if 0 is an eigenvalue of T_λ. Furthermore, the eigenvalues of T_λ are given, including multiplicity, by a sequence $\mu_1(\lambda)$, $\mu_2(\lambda)$, ... of functions on $\mathbb{R}$, which depend real-analytically on λ, and when λ is changed by some finite amount δ, the eigenvalues $\mu_\ell(\lambda)$ can each change by at most a constant which is independent of ℓ and only depends on δ (and, naturally, V).

So, we have established that non-scattering energies lead to zeros of $\mu_\ell(\lambda)$. Now the discreteness follows immediately from the observation that $Q_\lambda(u) > 0$ for all $\lambda \in \mathbb{R}_-$ and $u \in H_V \setminus 0$, as this means that none of the functions $\mu_\ell(\lambda)$ can vanish identically.

3 Corner scattering

Our second major topic is generalizing the non-existence of non-scattering energies to potentials with corners. The first such result only considers rectangular corners [5].

The novel approach introduced in [5] begins by assuming that a non-scattering energy exists with the intention to derive a contradiction. To illustrate the ideas, we assume that

$$\begin{cases} (-\Delta + V - \lambda)\, v = 0, \\ (-\Delta - \lambda)\, w = 0 \end{cases}$$

in $\mathbb{R}^2$, where $v, w \in B_2^*(\mathbb{R}^2)$ and $u = v - w \in \mathring{B}_2^*(\mathbb{R}^2)$. For simplicity, we assume here that the potential V is assumed to be supported in a closed sector $C \subseteq \mathbb{R}^2$ of angle smaller than π with vertex at the origin, and to be, say, smooth in C, compactly supported, and nonzero at the origin. Our paper [29] discusses more general situations.

The plan is to study the function w near the origin. As w is real-analytic, we may expand it as Taylor series in a neighbourhood of the origin, and pick the lowest degree nonzero terms, which form a harmonic homogeneous polynomial $H(x) \not\equiv 0$ of degree $N \geqslant 0$. The sought-for contradiction will come in the form $H(x) \equiv 0$.

It turns out that

$$\int_C V(x)\, w(x)\, \widetilde{w}(x)\, \mathrm{d}x = 0$$

for any solution $\widetilde{w} \in H^2_{\text{loc}}(x)$ to

$$(-\Delta + V - \lambda)\, \widetilde{w} = 0$$

in $\mathbb{R}^n$. A major component of [5] is constructing complex geometrical optics solutions of the form $\widetilde{w} = e^{-\rho \cdot x}(1 + \psi(x))$ for $\rho \in \mathbb{C}^n$ with $\rho \cdot \rho = 0$ and with ψ satisfying pleasant L^p estimates. In [5], such solutions were constructed in all dimensions $n \geq 2$ but for "polygonal" cones C (that is, the cross-section of the cone is a polygon). Our three dimensional result is used for circular cones, and so we base our CGO construction on certain L^p estimates from [21]. This argument gives sufficient estimates for $n \in \{2, 3\}$.

Thus, the CGO solutions are obtained here differently, and furthermore for ρ with $\rho \cdot \rho = \lambda$. Substituting these solutions to the the equality involving $\int_C$ gives, after some detailed estimations,

$$\int_C e^{-\rho \cdot x}\, H(x)\, \mathrm{d}x \ll |\rho|^{-N-2-\beta},$$

for some small $\beta \in \mathbb{R}_+$, as $|\rho| \longrightarrow \infty$, and we restrict to ρ such that, say, $\Re\rho \cdot x \geqslant \varepsilon > 0$ for all $x \in C$ for some fixed $\varepsilon \in \mathbb{R}_+$. On the other hand, by the homogeneity of $H(x)$, we have

$$\int_C e^{-\rho\cdot x}\, H(x)\,\mathrm{d}x = |\rho|^{-N-2} \int_C e^{-\rho/|\rho|\cdot x}\, H(x)\,\mathrm{d}x,$$

and this is compatible with the previous estimate only if

$$\int_C e^{-\rho\cdot x}\, H(x)\,\mathrm{d}x = 0$$

for certain $\rho \in \mathbb{C}^n$ with $\rho \cdot \rho = 0$ (as opposed to $\rho \cdot \rho = \lambda$).

At this point, our line of reasoning irreversibly departs from that of [5]. The main novelty in our paper [29] is based on moving here to polar coordinates, which turns out to reduce the dimension and lead to integrals over $C \cap S^{n-1}$. In two dimensions, we conclude that

$$\int_{C\cap S^1} (\rho\cdot\vartheta)^{-N-2}\, H(\vartheta)\,\mathrm{d}\vartheta = 0$$

for the same ρ as before, and where $\mathrm{d}\vartheta$ is the obvious measure on the unit circle $S^1 \subseteq \mathbb{R}^2$.

Since $H(x)$ was harmonic, it must be of the form

$$a\,(x_1 + ix_2)^N + b\,(x_1 - ix_2)^N$$

for some constants a and b. Now, choosing ρ suitably, the vanishing of the last integrals will lead, through some explicit calculations, to $a = b = 0$, establishing the desired contradiction.

In three dimensions, the same approach can largely be executed for circular cones. Now $H(x)$ will be a linear combination of spherical harmonics, and the vanishing of the coefficients will follow if certain integrals depending on the angle of the cone do not vanish. However, these integrals do not anymore seem to allow explicit evaluation. Yet, they depend analytically on the angle, and we are able to prove that none of them is identically zero, and this is enough for the conclusion that circular conical corners prohibit non-scattering energies except possibly for some at most countable set of exceptional angles.

References

[1] Adams, R.: *Capacity and compact imbeddings*, Journal of Mathematics and Mechanics, 19 (1970), 923–929.

[2] Adams, R., and J. Fournier: *Sobolev Spaces*, Pure and Applied Mathematics Series, Elsevier, 2003.

[3] Agmon, S.: *Spectral properties of Schrödinger operators and scattering theory*, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze, Serie IV, 2 (1975), 151–218.

[4] Agmon, S., and L. Hörmander: *Asymptotic properties of solutions of differential equations with simple characteristics*, Journal d'analyse mathématique, 30 (1976), 1–30.

[5] Blåsten, E., L. Päivärinta, and J. Sylvester: *Do corners always scatter?*, preprint at arXiv:1211.1848 (2012). To appear in Communications in Mathematical Physics.

[6] Cakoni, F., D. Gintides, and H. Haddar: *The existence of an infinite discrete set of transmission eigenvalues*, SIAM Journal on Mathematical Analysis, 42 (2010), 237–255.

[7] Cakoni, F., and H. Haddar: *Transmission eigenvalues in inverse scattering theory*, in [42], 529–578.

[8] Cakoni, F., and H. Haddar: *Transmission eigenvalues*, Inverse Problems, 29 (2013), 100201, 1–3.

[9] Colton, D., A. Kirsch, and L. Päivärinta: *Far field patterns for acoustic waves in an inhomogeneous medium*, SIAM Journal on Mathematical Analysis, 20 (1989), 1472–1482.

[10] Colton, D., and R. Kress: *Inverse Acoustic and Electromagnetic Scattering Theory*, Applied Mathematical Sciences, 93, Springer, 2013.

[11] Colton, D., and P. Monk: *The inverse scattering problem for time-harmonic acoustic waves in an inhomogeneous medium*, The Quarterly Journal of Mechanics and Applied Mathematics, 41 (1988), 97–125.

[12] Colton, D., L. Päivärinta and J. Sylvester: *The interior transmission problem*, Inverse Problems and Imaging, 1 (2007), 13–28.

[13] Grinevich, P. G., and S. V. Manakov: *The inverse scattering problem for the two-dimensional Schrödinger operator, the $\overline{\partial}$-method and non-linear equations*, Functional Analysis and its Applications, 20 (1986), 94–103.

[14] Grinevich, P. G., and R. G. Novikov: *Transparent potentials at fixed energy in dimension two. Fixed-energy dispersion relations for the fast decaying potentials*, Communications in Mathematical Physics, 174 (1995), 409–446.

[15] Hickmann, K.: *Interior transmission eigenvalue problem with refractive index having C^2-transition to the background medium*, Applicable Analysis, 91 (2012), 1675–1690.

[16] Hitrik, M., K. Krupchyk, P. Ola, and L. Päivärinta: *Transmission eigenvalues for operators with constant coefficients*, SIAM Journal on Mathematical Analysis, 2965–2986.

[17] Hörmander, L.: *Lower bounds at infinity for solutions of differential equations with constant coefficients*, Israel Journal of Mathematics, 16 (1973), 103–116.

[18] Hörmander, L.: *The Analysis of Linear Partial Differential Operators II: Differential Operators with Constant Coefficients*, Classics in Mathematics, Springer, 2005.

[19] Isozaki, H., and H. Morioka: *A Rellich type theorem for discrete Schrödinger operators*, arXiv:1208.4428 [math.SP].

[20] Kato, T.: *Perturbation Theory for Linear Operators*, Classics in Mathematics, Springer, 1995.

[21] Kenig, C. E., A. Ruiz, and C. D. Sogge: *Uniform Sobolev inequalities and unique continuation for second order constant coefficient differential operators*, Duke Mathematical Journal, 55 (1987), 329–347.

[22] Kirsch, A: *The denseness of the far field patterns for the transmission problem*, IMA Journal of Applied Mathematics, 37 (1986), 213–225.

[23] Lakshtanov, E., and B. Vainberg: *Applications of elliptic operator theory to the isotropic interior transmission eigenvalue problem*, Inverse Problems, 29 (2013), 104003, 1–19.

[24] Littman, W.: *Decay at infinity of solutions to partial differential equations with constant coefficients*, Transactions of the American Mathematical Society, 123 (1966), 449-459.

[25] Littman, W.: *Decay at infinity of solutions to partial differential equations; removal of the curvature assumption*, Israel Journal of Mathematics, 8 (1970), 403–407.

[26] Littman, W.: *Maximal rates of decay of solutions of partial differential equations*, Archive for Rational Mechanics and Analysis, 37 (1970), 11–20.

[27] Murata, M.: *Asymptotic behaviors at infinity of solutions to certain partial differential equations*, Journal of the Faculty of Science, the University of Tokyo, Section IA Mathematics, 23 (1976), 107–148.

[28] Newton, R. G.: *Construction of potentials from the phase shifts at fixed energy*, Journal of Mathematical Physics, 3 (1962), 75–82.

[29] Päivärinta, L., M. Salo, and E. V. Vesalainen: *Strictly convex corners scatter*, manuscript.

[30] Päivärinta, L., and J. Sylvester: *Transmission eigenvalues*, SIAM Journal on Mathematical Analysis, 40 (2008), 738–753.

[31] Regge, T.: *Introduction to complex orbital moments*, Il Nuovo Cimento, 14 (1959), 951–976.

[32] Rellich, F.: *Über das asymptotische Verhalten der Lösungen von $\Delta u + \lambda u = 0$ im unendlichen Gebieten*, Jahresbericht der Deutschen Mathematiker-Vereinigung, 53 (1943), 57–65.

[33] Robbiano, L.: *Spectral analysis on interior transmission eigenvalues*, Inverse Problems, 29 (2013), 104001, 1–28.

[34] Rudin, W.: *Real and Complex Analysis*, International Series in Pure and Applied Mathematics, McGraw-Hill, 1986.

[35] Sabatier, P. C.: *Asymptotic properties of the potentials in the inverse-scattering problem at fixed energy*, Journal of Mathematical Physics, 7 (1966), 1515–1531.

[36] Serov, V.: *Transmission eigenvalues for non-regular cases*, Communications in Mathematical Analysis, 14 (2013), 129–142.

[37] Serov, V., and J. Sylvester: *Transmission eigenvalues for degenerate and singular cases*, Inverse Problems, 28 (2012), 065004, 1–8.

[38] Shaban, W., and B. Vainberg: *Radiation conditions for the difference Schrödinger operators*, Applicable Analysis, 80 (2001), 525–556.

[39] Sylvester, J.: *Discreteness of transmission eigenvalues via upper triangular compact operators*, SIAM Journal on Mathematical Analysis, 44 (2012), 341–354.

[40] Sylvester, J., and G. Uhlmann: *A global uniqueness theorem for an inverse boundary value problem*, Annals of Mathematics, 125 (1987), 153–169.

[41] Trèves, F.: *Differential polynomials and decay at infinity*, Bulletin of the American Mathematical Society (New Series), 66 (1960), 184–186.

[42] Uhlmann, G. (editor): *Inverse Problems and Applications: Inside Out II*, MSRI Publications 60, Cambridge University Press, 2013.

[43] Vekua, I. N.: *Metaharmonic functions*, Trudy Tbilisskogo matematicheskogo instituta, 12 (1943), 105–174.

[44] Vesalainen, E. V.: *Transmission eigenvalues for a class of non-compactly supported potentials*, Inverse Problems, 29 (2013), 104006, 1–11.

[45] Vesalainen, E. V.: *Rellich type theorems for unbounded domains*, submitted.

I. Transmission Eigenvalues for a Class of Non-Compactly Supported Potentials

Esa V. Vesalainen

Abstract

Let $\Omega \subseteq \mathbb{R}^n$ be a non-empty open set for which the Sobolev embedding $H_0^2(\Omega) \longrightarrow L^2(\Omega)$ is compact, and let $V \in L^\infty(\Omega)$ be a potential taking only positive real values and satisfying the asymptotics $V(\cdot) \asymp \langle\cdot\rangle^{-\alpha}$ for some $\alpha \in]3, \infty[$. We establish the discreteness of the set of real transmission eigenvalues for both Schrödinger and Helmholtz scattering with these potentials.

1 Introduction

1.1 Non-scattering energies and non-scattering wavenumbers

We shall be concerned with the interior transmission problem for the Schrödinger and Helmholtz equations. Inverse scattering theory, and the study of the linear sampling method and the factorization method in particular, gives rise to the study of non-scattering energies. These are energies $\lambda \in \mathbb{R}_+$ for which there exists a non-zero incoming wave which does not scatter in the sense that the corresponding scattered wave has a vanishing main term in its asymptotic expansion. In the case of the Schrödinger equation with a short-range potential $V \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ this ultimately means that the system

$$\begin{cases} (-\Delta + V - \lambda)\, v = 0, \\ (-\Delta - \lambda)\, w = 0 \end{cases}$$

has a solution $v, w \in B_2^* \setminus 0$ where the two functions are connected by the asymptotic condition $v - w \in \mathring{B}_2^*$. The Helmholtz case is otherwise the same, except that the perturbed equation for v is

$$(-\Delta + \lambda V - \lambda)\, v = 0,$$

and the term non-scattering wavenumber is more appropriate.

Here the solutions are taken from the function spaces

$$B_2^* = \left\{ u \in B^* \,\middle|\, \partial^\gamma u \in B^*, \forall\, |\gamma| \leqslant 2 \right\},$$

and

$$\mathring{B}_2^* = \left\{ u \in \mathring{B}^* \,\middle|\, \partial^\gamma u \in \mathring{B}^*, \forall\, |\gamma| \leqslant 2 \right\},$$

where B^* consists of those functions $u \in L^2_{\text{loc}}(\mathbb{R}^n)$ for which

$$\sup_{R>1} \frac{1}{R} \int_{B(0,R)} |u|^2 < \infty,$$

where $B(0,R)$ is the ball of radius R centered at the origin, and $\mathring{B}^*$ consists of those functions $u \in B^*$ for which

$$\frac{1}{R} \int_{B(0,R)} |u|^2 \longrightarrow 0$$

as $R \longrightarrow \infty$. A function $V \in L^2_{\text{loc}}(\mathbb{R}^n)$ is a short-range potential for instance when $V(\cdot) \ll \langle\cdot\rangle^{-\alpha}$ in $\mathbb{R}^n$ for some $\alpha \in \,]1, \infty[$. For a presentation of short-range scattering theory, see e.g. Chapter XIV of [11] and the first sections of [15].

1.2 Interior transmission eigenvalues

If the potential V vanishes outside a suitable bounded domain Ω, then the functions v and w coincide outside Ω (by Rellich's lemma and unique continuation) and we are left with a solution to the problem

$$\begin{cases} (-\Delta + V - \lambda)\, v = 0, \\ (-\Delta - \lambda)\, w = 0, \end{cases}$$

where v and w are to be in $H^2_{\text{loc}}(\Omega) \cap L^2(\Omega)$ and to satisfy the boundary conditions $v - w \in H^2_0(\Omega)$. This problem is the interior transmission problem for Ω and V.

Typical first steps in the study of the interior transmission problem are the finite multiplicity of transmission eigenvalues, the discreteness of the set of transmission eigenvalues, and the existence of infinitely many transmission eigenvalues.

1.3 The purpose and the motivation of this work

Since scattering theory does not really care about the support of V, it is natural to ask whether the study of the interior transmission problem can be carried over to non-compact supports. A particularly strong motivation for studying this is that, metaphorically speaking, non-scattering energies are transmission eigenvalues for the domain $\Omega = \mathbb{R}^n$. In this particular case, the combinations of the techniques of short-range scattering theory and interior transmission eigenvalue problems might allow a new approach to directly deal with non-scattering energies.

One particular question which might be approached in this way is the existence of non-scattering energies. For compactly supported radial scatterers, there are always infinitely many of them as in that case the non-scattering energies coincide with the transmission eigenvalues. On the other hand, it was recently shown by Blåsten, Päivärinta and Sylvester [4] that for a large class of potentials there are no non-scattering energies. It is not yet known if non-scattering energies can exist for non-radial potentials.

In the following we shall take first steps into the direction of non-compact supports by considering interior transmission eigenvalues for non-compact Ω which are nearly compact in the sense that they have a suitable compact Sobolev embedding, and for potentials V taking only positive real values and having a certain kind of asymptotic behaviour. For these potentials, we shall prove the basic discreteness result. This is done by proving the basic discreteness and existence results for a closely connected fourth-order equation. The more usual case of bounded Ω with a positive real-valued potential, which is bounded and bounded away from zero, is covered as a special case, including the corresponding existence result for Helmholtz transmission eigenvalues.

It should be noted that this discreteness result would imply the discreteness for the corresponding non-scattering energies if a conclusion analogous to that of Rellich's lemma could be somehow obtained. It seems that there are no known generalizations of Rellich's lemma to non-compact domains, but such generalizations might exist. We intend to return to this topic in the future.

1.4 A few words on the preceding work

The interior transmission problem first appeared in the papers of Kirsch [13], and Colton and Monk [8]. The first papers considered radial potentials and discreteness for general potentials, see e.g. the survey [9] of Colton, Päivärinta and Sylvester. The first general existence result was obtained by Päivärinta and Sylvester [15], and later Cakoni, Gintides and Haddar [6] proved the first general result on existence of infinitely many transmission eigenvalues.

It should be noted that the methods in the papers of Sylvester [19], Lakshtanov and Vainberg [14] and Robbiano [16] are able to handle compactly supported potentials with fairly arbitrary behaviour inside the domain. I.e. the main assumptions only deal with the behaviour of the potentials in a neighbourhood of the boundary.

It is clear that we can not give here an exhaustive list of previous results and references. For a recent survey on the topic, we recommend the article [7] by Cakoni and Haddar.

The main result of this paper and its proof are in their spirit closest to the work of Hickmann [10], Serov and Sylvester [18], and Serov [17], who proved discreteness and existence results in compact domains for potentials exhibiting well controlled degenerate or singular behaviour at the boundary of the domain using quadratic forms, suitable weighted spaces and Hardy-type inequalities.

1.5 On notation

We shall employ the standard asymptotic notation. Given two complex functions A and B defined on some set Ω, the relation $A \ll B$ means that $|A| \leqslant C\,|B|$ in Ω for some positive real constant C. The relation $A \asymp B$ means that both $A \ll B$ and $A \gg B$, and $A \gg B$ means the same as $B \ll A$. We do not insist on the implicit constants being computable.

When the letter ε appears in various exponents, it denotes an arbitrarily small, and also sufficiently small, positive real constant, which usually changes its value from one occurrence to the next. The usage of this notational device should be rather transparent.

For a vector $\xi \in \mathbb{R}^n$, we let $\langle \xi \rangle$ denote $\sqrt{1 + |\xi|^2}$, as usual.

2 The main theorems

We fix the dimension $n \in \mathbb{Z}_+$ of the ambient Euclidean space for the entire text, and all implicit constants are allowed to depend on it. Let $\Omega \subseteq \mathbb{R}^n$ be an open set for which the Sobolev embedding $H_0^2(\Omega) \longrightarrow L^2(\Omega)$ is compact, and let $V \in L^2_{\text{loc}}(\Omega)$ be a potential taking only positive real values and satisfying the asymptotics $V(\cdot) \asymp \langle \cdot \rangle^{-\alpha}$ for some $\alpha \in \,]3, \infty[$.

For sufficient conditions on Ω guaranteeing the compact embedding, see the chapter 6 of [2], in particular Theorems 6.16 and 6.19, or the original article [1]. The conditions are somewhat technical and therefore we do not reproduce them here. However, when $n \leqslant 3$, one has the pleasant characterization: the embedding $H_0^2(\Omega) \longrightarrow L^2(\Omega)$ is compact if and only if Ω does not contain infinitely many pairwise disjoint balls which are all of the same size (see remarks 6.17.3, 6.9 and 6.11 in [2]).

The theorems below cover as a special case bounded domains Ω with potentials V, which take only positive real values, and which are bounded and bounded away from zero.

In our setting, transmission eigenvalues for the Schrödinger equation are defined to be those complex numbers λ for which there exist functions

$$v, w \in \left\{ u \in H^2_{\text{loc}}(\Omega) \;\middle|\; \widetilde{u} \in B^* \right\} \setminus 0$$

solving the equations

$$(-\Delta + V - \lambda)\, v = 0, \qquad (-\Delta - \lambda)\, w = 0$$

in Ω, and connected by the asymptotic relation and boundary conditions

$$v - w \in \overset{\circ}{B}{}^*_2(\Omega) = \left\{ u \in H^2_{\text{loc}}(\Omega) \;\middle|\; \widetilde{u} \in B^*_2(\mathbb{R}^n) \right\},$$

where $\widetilde{u} \colon \mathbb{R}^n \longrightarrow \mathbb{C}$ coincides with u in Ω and vanishes identically elsewhere. It does no harm to occasionally identify u with its zero extension $\widetilde{u}$.

The multiplicity of a transmission eigenvalue is defined as the dimension of the vector space of pairs of functions $\langle v, w \rangle$ solving the above problem.

We shall only consider real transmission eigenvalues and this is a genuine restriction (as was first shown by F. Cakoni, D. Colton and D. Gintides [5]).

Our main theorem is

Theorem 1. *The set of positive real transmission eigenvalues for the Schrödinger equation is a discrete subset of* $[0, \infty[$, *and each of its elements is of finite multiplicity.*

For the Helmholtz equation the perturbed equation for v is

$$(-\Delta + \lambda V - \lambda)\, v = 0,$$

and one excludes the uninteresting value $\lambda = 0$, but otherwise everything else is the same. In particular, we have

Theorem 2. *The set of positive real transmission eigenvalues for the Helmholtz equation is a discrete subset of* $[0, \infty[$, *and each of its elements is of finite multiplicity.*

3 Proof of Theorem 1

3.1 Reduction to a fourth-order equation

The first step in the proof is writing the transmission eigenvalue problem as a single fourth-order partial differential equation. This idea is rather standard and is the basis for many discreteness and existence proofs in the literature. The non-vanishing of V is rather essential here.

We shall handle the operator in the fourth-order equation using quadratic forms, and this will require a shift from the B^*-based spaces to certain weighted L^2-based spaces. The role of the ambient space will be played by L_V, a space which we define to consist of those L^2_{loc}-functions in Ω whose zero extensions belong to Agmon's weighted space

$$L^{2,\alpha/2}(\mathbb{R}^n) = \left\{ u \in L^2_{\mathrm{loc}}(\mathbb{R}^n) \,\middle|\, \langle\cdot\rangle^{\alpha/2} u \in L^2(\mathbb{R}^n) \right\},$$

which of course is a Hilbert space with the right weighted L^2-norm.

The quadratic form domain will be H_V, a space which we define to consist of those L^2_{loc}-functions in Ω whose zero extensions belong to Agmon's weighted space

$$H_{2,\alpha/2}(\mathbb{R}^n) = \left\{ u \in L^2_{\mathrm{loc}}(\mathbb{R}^n) \,\middle|\, \partial^\gamma u \in L^{2,\alpha/2}(\mathbb{R}^n), \forall\, |\gamma| \leqslant 2 \right\}.$$

The space H_V is Hilbert when equipped with the restriction of the $H_{2,\alpha/2}$-norm.

We point out that H_V embeds compactly into L_V. It is easy to split this embedding into three parts

$$H_V \longrightarrow H^2_0(\Omega) \longrightarrow L^2(\Omega) \longrightarrow L_V,$$

where the middle one is the obvious embedding, which is assumed to be compact, and the first and the last mappings are multiplications by $\langle\cdot\rangle^{\alpha/2}$ and $\langle\cdot\rangle^{-\alpha/2}$, respectively.

Now we are ready to state the transition to a fourth-order equation:

Lemma 1. *If a positive real number λ is a transmission eigenvalue then there exists a function $u \in H_V \setminus 0$ solving the equation*

$$(-\Delta + V - \lambda)\,\frac{1}{V}\,(-\Delta - \lambda)\,u = 0 \tag{1}$$

in Ω in the sense of distributions. Furthermore, this transition retains multiplicities.

If v and w solve the transmission eigenvalue problem, then it is a matter of simple calculation to see that $u = v - w$ solves the fourth-order equation. It only remains to see that $u \in \overset{\circ}{B}{}_2^*(\Omega)$ corresponding to a transmission eigenvalue necessarily belongs to H_V. This follows from the observation that

$$(-\Delta - \lambda)\, u = -Vv,$$

not only in Ω but also in $\mathbb{R}^n$. The function $\langle\cdot\rangle^{\alpha-1-\varepsilon}\, V$ has enough decay to be a short-range potential, and so $\langle\cdot\rangle^{\alpha-1-\varepsilon}\, Vv \in B$. Now, by a basic inequality in short-range scattering theory (see e.g. Theorem 14.3.7 in [11]),

$$\langle\cdot\rangle^{\alpha-1-\varepsilon}\, \partial^\gamma u \in B^*,$$

for $|\gamma| \leqslant 2$. It is then easy to check that

$$\langle\cdot\rangle^{\alpha-3/2-\varepsilon}\, \partial^\gamma u \in L^2(\mathbb{R}^n),$$

again for $|\gamma| \leqslant 2$, which in turn implies

$$\langle\cdot\rangle^{\alpha/2}\, \partial^\gamma u \in L^2(\mathbb{R}^n),$$

since $\alpha > 3$.

From now on, we focus on studying the spectral properties of the fourth-order equation (1). In particular, we shall establish a discreteness result and a conditional existence result. The discreteness result, together with Lemma 1, implies Theorem 1. The hypothesis required for the general existence result concerns the existence for suitable simple cases.

Hypothesis 1. *For any ball B in $\mathbb{R}^n$ and any constant potential $V_0 \in \mathbb{R}_+$, there exists a Schrödinger transmission eigenvalue.*

We do not know whether this hypothesis is true.

By inspecting the conditional existence proof (which will be given in Section 3.6), we see that there is some freedom in the formulation of the hypothesis. For example, we only need the existence for a sequence of balls and positive constant potentials, where both the radii of the balls and the potentials tend to zero. Furthermore, balls could be replaced by any domains whose diameters shrink to zero, and the potentials do not have to be constant, as long as their L^∞-norms tend to zero, and they are positive and bounded away from zero.

Also, it should be noted, that by considering radial functions (see Section 4) one can prove that there are transmission eigenvalues for any ball, provided that V_0 is sufficiently large. From this the approach of Section 3.6 will give unconditional existence of eigenvalues (not necessarily infinitely many), provided that the potential V is sufficiently large in some balls in Ω.

Theorem 3. *The set of real numbers λ for which the equation (1) has a non-trivial H_V-solution is a discrete subset of $[0, \infty[$. For each such λ the space of solutions is finite dimensional. Furthermore, if the Hypothesis 1 holds, the set of such real numbers λ is infinite.*

For a bounded Ω the H_V-solutions that can be conditionnally obtained by this theorem give rise to transmission eigenvalues, respecting multiplicities, and we get the existence of infinitely many transmission eigenvalues. Unfortunately, in the unbounded case this does not work. The obstacle is that the solutions to the fourth-order equation belong to weighted spaces which essentially guarantee that division by $\sqrt{V}$ is a reasonably good operation, whereas in order to get from the fourth-order equation back to the interior transmission problem one needs to divide by V, an operation genuinely worse than division by $\sqrt{V}$, and there seems to be no way of guaranteeing that the asymptotic behaviour of the apparent transmission eigenfunction pair is sufficiently good.

3.2 The quadratic forms

We will handle the operator on the left-hand side of (1) via quadratic forms, and for this purpose we define for each $\lambda \in \mathbb{C}$ the quadratic form

$$Q_\lambda = u \longmapsto \left\langle (-\Delta + V - \overline{\lambda})\, u \,\middle|\, \frac{1}{V}\, (-\Delta - \lambda)\, u \right\rangle : H_V \longrightarrow \mathbb{C},$$

where the L^2-inner product is linear in the second argument. Instead of considering Q_λ as a quadratic form in $L^2(\Omega)$, we shall consider it in the weighted L^2-space L_V. The idea of using weighted L^2-spaces as the ambient Hilbert spaces, in order to handle degenerate or even singular potentials in the case of a bounded domain, has been used in the papers [10], [18] and [17], where the weight is a power of distance to the boundary of the domain.

It turns out that the family $\langle Q_\lambda \rangle_{\lambda \in \mathbb{C}}$ has the pleasant properties enumerated in the theorem below. An excellent reference for the basic theory of quadratic forms and analytic perturbation theory used is the book by Kato [12], in particular its Chapters VI and VII. More detailed references will be given in the course of the proofs of the statements.

Theorem 4.

1. *The quadratic forms Q_λ form an entire self-adjoint analytic family of forms of type (a) with compact resolvent, and therefore gives rise to a family of operators T_λ, which is an entire self-adjoint analytic family of operators of type (B) with compact resolvent.*
2. *Furthermore, there exists a sequence $\langle \mu_\nu(\cdot) \rangle_{\nu=1}^{\infty}$ of real-analytic functions $\mu_\nu(\cdot) \colon \mathbb{R} \longrightarrow \mathbb{R}$ such that, for real λ, the spectrum of T_λ, which consists*

of a discrete set of real eigenvalues of finite multiplicity, consists of $\mu_1(\lambda)$, $\mu_2(\lambda)$, $\ldots$, *including multiplicity.*

3. *In addition, for any given* $T \in \mathbb{R}_+$, *there exists constant* $c \in \mathbb{R}_+$ *such that*

$$\left|\mu_\nu(\lambda) - \mu_\nu(0)\right| \ll_T e^{c|\lambda|} - 1$$

for all $\lambda \in [-T, T]$ *and each* $\nu \in \mathbb{Z}_+$.

4. *The pairs* $\langle \lambda, u \rangle \in \mathbb{R} \times H_V$ *for which (1) holds, are in bijective correspondence with the pairs* $\langle \nu, \lambda \rangle \in \mathbb{Z}_+ \times \mathbb{R}$ *for which* $\mu_\nu(\lambda) = 0$.

5. *If Hypothesis 1 holds, then there are infinitely many such pairs* $\langle \lambda, u \rangle$.

The discreteness result follows easily from these properties of Q_λ. It is obvious that zero is not an eigenvalue of T_λ for any negative real λ, as $Q_\lambda(u) > 0$ for all non-zero functions $u \in \operatorname{Dom} Q_\lambda$. Hence none of the functions $\mu_\nu(\cdot)$ can vanish identically, so that the set of zeroes of each of them is discrete. Why the union of the zero sets can not have an accumulation point follows immediately from the third statement above, which says that the functions $\mu_\nu(\cdot)$ change their values uniformly locally exponentially. That is, when the value of λ changes by a finite amount, only finitely many $\mu_\nu(\cdot)$ will have enough time to drop to zero.

The second statement follows immediately from a basic result in the perturbation theory of linear operators, once the first has been proven; for this see [12, rem. VII.4.22, p. 408] and the backwards references. The third statement comes from theorem VII.4.21 [12, p. 408]. The fourth and fifth statements will be consequences of the mini-max principle, but will be given only after the first one has been dealt with.

We remark that the proof of the fifth statement only requires continuity of the family $\langle \mu_\nu(\cdot) \rangle_{\nu=1}^{\infty}$, which can be proved using the mini-max principle with no reference to non-real values of λ (see e.g. the proof of Lemma 12 in [15]). The observation that these eigenvalues depend real-analytically on λ seems to be new.

3.3 A weighted inequality

The proof of closedness of Q_λ will depend on the following weighted inequality.

Lemma 2. *Let* $K \subseteq \mathbb{C}$ *be compact, and let* $s \in \mathbb{R}$. *Then*

$$\left\|\langle\cdot\rangle^s u\right\| + \left\|\langle\cdot\rangle^s \nabla u\right\| + \left\|\langle\cdot\rangle^s \nabla \otimes \nabla u\right\| \ll_{K,s} \left\|\langle\cdot\rangle^s (-\Delta - \lambda) u\right\| + \left\|\langle\cdot\rangle^s u\right\|$$

for all $u \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$ *and* $\lambda \in K$.

Here and elsewhere, given an expression $E(\cdot)$, we use the short-hand notations $E(\nabla)$ and $E(\nabla \otimes \nabla)$ for $\sum_{|\alpha|=1} E(\partial^\alpha)$ and $\sum_{|\alpha|=2} E(\partial^\alpha)$, respectively. When necessary, we shall use other similar short-hands whose meaning will be clear.

Proof of Lemma 2. The following argument is an adaptation of the proof of Lemma A.3 of [3, p. 206]. Since

$$\langle \cdot \rangle^4 \ll_K \left| 4\pi^2 \left| \cdot \right|^2 - \lambda \right|^2 + 1,$$

multiplication by $\left| \widehat{u} \right|^2$ and integration over $\mathbb{R}^n$ gives

$$\|u\| + \|\nabla u\| + \|\nabla \otimes \nabla u\| \ll_K \|(-\Delta - \lambda)\, u\| + \|u\|.$$

In order to introduce weights, we observe that for $\varepsilon \in \,]0, 1]$,

$$\langle \cdot \rangle \asymp_{\varepsilon, s} \langle \varepsilon \cdot \rangle,$$

and that

$$\partial^\alpha \langle \varepsilon \cdot \rangle^s \ll_{\varepsilon, s} \langle \varepsilon \cdot \rangle^s.$$

Now Leibniz's rule, the weightless inequality and the triangle inequality give

$$\begin{aligned}
&\|\langle \varepsilon \cdot \rangle^s u\| + \|\langle \varepsilon \cdot \rangle^s \nabla u\| + \|\langle \varepsilon \cdot \rangle^s \nabla \otimes \nabla u\| \\
&\quad \ll \|\langle \varepsilon \cdot \rangle^s u\| + \|\nabla(\langle \varepsilon \cdot \rangle^s u)\| + \|\nabla \otimes \nabla(\langle \varepsilon \cdot \rangle^s u)\| \\
&\qquad + \|(\nabla \langle \varepsilon \cdot \rangle^s)\, u\| + \|(\nabla \langle \varepsilon \cdot \rangle^s) \otimes \nabla u\| + \|(\nabla \otimes \nabla \langle \varepsilon \cdot \rangle^s)\, u\| \\
&\quad \ll_K \|(-\Delta - \lambda)\,(\langle \varepsilon \cdot \rangle^s u)\| + \|\langle \varepsilon \cdot \rangle^s u\| \\
&\qquad + \|(\nabla \langle \varepsilon \cdot \rangle^s)\, u\| + \|(\nabla \langle \varepsilon \cdot \rangle^s) \otimes \nabla u\| + \|(\nabla \otimes \nabla \langle \varepsilon \cdot \rangle^s)\, u\| \\
&\quad \ll \|\langle \varepsilon \cdot \rangle^s (-\Delta - \lambda)\, u\| + \|(\nabla \langle \varepsilon \cdot \rangle^s) \cdot \nabla u\| + \|(\Delta \langle \varepsilon \cdot \rangle^s)\, u\| + \|\langle \varepsilon \cdot \rangle^s u\| \\
&\qquad + \|(\nabla \langle \varepsilon \cdot \rangle^s)\, u\| + \|(\nabla \langle \varepsilon \cdot \rangle^s) \otimes \nabla u\| + \|(\nabla \otimes \nabla \langle \varepsilon \cdot \rangle^s)\, u\| \\
&\quad \ll_s \|\langle \varepsilon \cdot \rangle^s (-\Delta - \lambda)\, u\| + \varepsilon \|\langle \varepsilon \cdot \rangle^s \nabla u\| + \varepsilon^2 \|\langle \varepsilon \cdot \rangle^s u\| + \|\langle \varepsilon \cdot \rangle^s u\| \\
&\qquad + \varepsilon \|\langle \varepsilon \cdot \rangle^s u\| + \varepsilon \|\langle \varepsilon \cdot \rangle^s \nabla u\| + \varepsilon^2 \|\langle \varepsilon \cdot \rangle^s u\| \\
&\quad \ll \|\langle \varepsilon \cdot \rangle^s (-\Delta - \lambda)\, u\| + \|\langle \varepsilon \cdot \rangle^s u\| + \varepsilon \|\langle \varepsilon \cdot \rangle^s \nabla u\|.
\end{aligned}$$

Choosing a sufficiently small ε, subject to the choices of K and s, allows us to eliminate the first-order term from the right-hand side, giving the weighted version of the desired inequality.

3.4 Q_λ is a good self-adjoint family

The fact that the family $\langle T_\lambda \rangle_{\lambda \in \mathbb{C}}$ is a self-adjoint analytic family of type (B) with compact resolvent will follow from a number of different results in the aforementioned book [12].

If $\langle Q_\lambda \rangle$ form a self-adjoint analytic family of quadratic forms of type (a), then for each $\lambda \in \mathbb{C}$, there corresponds a unique closed linear operator T_λ; since Q_λ is densely defined, sectorial and closed (as will be shown later), the unique existence of T_λ is given by [12, thm. VI.2.1, p. 322], and the operator T_λ is furthermore m-sectorial.

The theorem VII.4.2 [12, p. 395] then says that $\langle T_\lambda \rangle$ is an analytic family of operators (in the sense of Kato). Since

$$\operatorname{Dom} T_\lambda \subseteq \operatorname{Dom} Q_\lambda = H_V,$$

and H_V embeds compactly into L_V, the family $\langle T_\lambda \rangle$ has compact resolvent. Finally, the family is self-adjoint, i.e. $T_\lambda^* = T_{\overline{\lambda}}$, since $\overline{Q_\lambda} = Q_{\overline{\lambda}}$ for all $\lambda \in \mathbb{C}$. This follows from theorem VI.2.5 [12, p. 323]. In particular, T_λ is a self-adjoint operator with compact resolvent for real λ.

Thus it remains to prove that $\langle Q_\lambda \rangle$ is an analytic family of type (a). By definition, this entails checking that

- Each Q_λ is sectorial and closed, and $\operatorname{Dom} Q_\lambda$ is independent of λ; and
- $Q_\lambda(u)$ is an entire function of λ for any fixed $u \in \operatorname{Dom} Q_\lambda$.

The latter condition is obviously satisfied as $Q_\lambda(u)$ is, in fact, a second degree polynomial in λ. That $\operatorname{Dom} Q_\lambda$ is independent of λ is also obvious here, because the domain is simply H_V. So it only remains to prove that each Q_λ is sectorial and closed.

That Q_λ is sectorial simply means that the set $Q_\lambda[\{u \in H_V \mid \|u\|_{L_V} = 1\}]$ is contained in a sector-shaped set of the form

$$\{z \in \mathbb{C} \mid \arg(z - z_0) \leqslant \vartheta\}$$

for some fixed $z_0 \in \mathbb{C}$ and $\vartheta \in \left[0, \frac{\pi}{2}\right[$. This sectoriality condition can be established by the usual elementary arguments; see e.g. Example 1.7 in [12, p. 312].

That Q_λ is closed follows now from the fact that, by the weighted inequality proved above, the $H_{2,\alpha/2}$-norm and the norm arising from Q_λ, given by the expression

$$\sqrt{\Re Q_\lambda(\cdot) + (1 + \lambda)\| \cdot \|_{L_V}^2},$$

are comparable on $C_c^\infty(\Omega)$ and therefore the domain of Q_λ is really just the closure of test functions of Ω in the right norm.

3.5 The bijective correspondence between $\langle \lambda, u \rangle$ and $\langle \nu, \lambda \rangle$

If zero is an eigenvalue of T_λ with an eigenfunction $u \in H_V$, then clearly $Q_\lambda(v, u) = 0$ for all $v \in C_c^\infty(\Omega)$, and u is a non-trivial solution to the equation (1).

The other direction is only slightly more challenging to establish. Suppose that for $\lambda \in \mathbb{R}$ the equation (1) has a non-trivial space of solutions in H_V of dimension N. Then Q_λ vanishes in some subspace $Y \subseteq H_V$ of dimension N, and in fact, $Q_\lambda(u, v) = 0$ for all $v \in H_V$ and $u \in Y$. Our goal is to prove that zero is an eigenvalue of T_λ of multiplicity at least N using the mini-max principle. Let the spectrum of T_λ be

$$\mu_1 \leqslant \mu_2 \leqslant \mu_3 \leqslant \dots .$$

The space X corresponding to the negative eigenvalues of T_λ is finite dimensional, say of dimension $m \geqslant 0$. Now the restriction

$$T|_{X^\perp} : X^\perp \cap \operatorname{Dom} T_\lambda \longrightarrow X^\perp$$

is again a self-adjoint operator with compact resolvent and no negative eigenvalues. The eigenvalues μ_{m+1}, μ_{m+2}, …, μ_{m+N} all have to be non-negative.

Conversely, μ_{m+N} is at most

$$\begin{aligned} &\max \left\{ Q_\lambda(f) \,\middle|\, f \in \operatorname{span}\{X, Y\}, \|f\|_{L_V} = 1 \right\} \\ = &\max \left\{ (Q_\lambda(g) + 2\Re Q_\lambda(g, h) + Q_\lambda(h)) \,\middle|\, g \in X, h \in Y, \|g + h\|_{L_V} = 1 \right\}, \end{aligned}$$

and in the expression $(\dots)$ the first term is certainly $\leqslant 0$ and the remaining terms vanish. Thus $\mu_{m+1} \leqslant \mu_{m+2} \leqslant \dots \leqslant \mu_{m+N} \leqslant 0$ and we are done.

3.6 The conditional infinitude of zeroes of $\mu_\nu(\cdot)$

Next we shall prove that, under Hypothesis 1, for arbitrarily large positive integers N, there exists at least N pairs $\langle \nu, \lambda \rangle \in \mathbb{Z}_+ \times \mathbb{R}$ satisfying $\mu_\nu(\lambda) = 0$. This is achieved by comparison to the simpler domains with constant potentials for which the existence of a single transmission eigenvalue is guaranteed by Hypothesis 1.

We choose N small balls B_1, B_2, …, B_N, whose closures are in Ω and pairwise disjoint, and consider on them a constant potential $V_0 \in \mathbb{R}_+$ such that $V_0 \leqslant V$ in $B_1 \cup B_2 \cup \ldots \cup B_N$, and such that there is a number $\lambda \in \mathbb{R}_+$ which is a transmission eigenvalue for each of the balls. The above theorem guarantees the existence of such a small V_0.

The H_0^2-spaces of the balls naturally embed into H_V by taking zero extensions of their elements. Denote by $H(N)$ the closed subspace spanned by the images of the differences of the transmission eigenfunction pairs of V_0 in the small balls. This space has dimension at least N.

Now the quadratic form $\widetilde{Q}_\lambda$ corresponding to the constant potential V_0 in Ω is basically the Q_λ defined above, but with $1/V$ replaced by $1/V_0$. In particular, we have the inequality $Q_\lambda \leqslant \widetilde{Q}_\lambda = 0$ in $H(N)$. (The domain of $\widetilde{Q}_\lambda$ can be chosen to be anything reasonable that contains $H(N)$ as we only need this non-positivity inequality.)

The eigenvalues of T_κ are positive for $\kappa \in \mathbb{R}_-$, but by the mini-max principle, at least N of the eigenvalues of T_λ are non-positive. Therefore the functions $\mu_n(\cdot)$ must have at least N zeroes in the interval $[0, \lambda]$.

4 Some remarks on the Helmholtz case

Everything we do works for the Helmholtz equation with modest modifications. The fourth-order equation (1) should be replaced by

$$(-\Delta + \lambda V - \lambda)\,\frac{1}{V}\,(-\Delta - \lambda)\,u = 0, \tag{2}$$

and the quadratic forms should be redefined accordingly. The spectral properties will in this case be slightly better than in the Schrödinger case:

Theorem 5. *The set of positive real numbers λ for which the equation (2) has a non-trivial H_V-solution is an infinite discrete subset of $[0, \infty[$, and for each such λ the space of solutions is finite dimensional. Furthermore, the number of such λ not exceeding $x \in \mathbb{R}_+$, counting multiplicities, is $\gg x^{n/2}$ as $x \longrightarrow \infty$.*

The unconditional existence proof depends on

Theorem 6. *For a ball B in $\mathbb{R}^n$, and an arbitrarily small constant potential $c \in \mathbb{R}_+$, there exist infinitely many positive real Helmholtz transmission eigenvalues.*

This is a special case of a much more general theorem on existence for radial potentials, a proof of which in three dimensions may be found in [9, p. 16]. For constant potentials, the proof simplifies nicely, and though it seems that there is no n-dimensional proof in the literature, the 3-dimensional proof generalizes easily: the crucial difference is that $j_0(r)$ must be replaced by $r^{(2-n)/2}\,J_{(n-2)/2}(r)$.

Now we do not immediately see that $\mu_n(\lambda) > 0$ for negative λ and each $n \in \mathbb{Z}_+$. Instead, we observe easily that $\mu_n(0) > 0$ for each n: If $Q_0(u) = 0$, then $\Delta u \equiv 0$, implying that $\nabla u \equiv 0$, and therefore u must vanish.

The asymptotic lower bound $\gg x^{n/2}$ for the number of zeroes of $\mu_\nu(\cdot)$ not exceeding a large positive real number x follows from the fact that transmission eigenvalues for the Helmholtz equation scale under dilations like Dirichlet eigenvalues.

More precisely, let us look at a ball B whose closure is contained in Ω, and let $V_0 \in \mathbb{R}_+$ be so small that $V_0 \leqslant V(\cdot)$ in B. Now there exists a transmission eigenvalue λ for B and the constant potential V_0.

Given any $\varepsilon \in \mathbb{R}_+$, it is easy to see that the number $\frac{\lambda}{\varepsilon^2}$ is a transmission eigenvalue for any translate of εB with the constant potential V_0.

Let $x \in \mathbb{R}_+$, and choose $\varepsilon = \lambda^{1/2} x^{-1/2}$. Now the number $\frac{\lambda}{\varepsilon^2} = x$ is a transmission eigenvalue for any translate of εB with the constant potential V_0, and we can pack $\gg \varepsilon^{-n} \gg_\lambda x^{n/2}$ such translates inside B so that no two of them intersect, provided that x is large enough. These will correspond to the balls B_1, B_2, $\ldots$, B_N of section 3.6. Now we finish the proof in the same way as in section 3.6 and obtain $\gg_\lambda x^{n/2}$ zeroes not exceeding x.

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References

[1] Adams, R.: *Capacity and compact imbeddings*, Journal of Mathematics and Mechanics, 19 (1970), 923–929.

[2] Adams, R., and J. Fournier: *Sobolev Spaces*, Pure and Applied Mathematics Series, Elsevier, 2003.

[3] Agmon, S.: *Spectral properties of Schrödinger operators and scattering theory*, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze, Serie IV, 2 (1975), 151–218.

[4] Blåsten, E., L. Päivärinta, and J. Sylvester: *Do corners always scatter?*, preprint at arXiv:1211.1848 (2012).

[5] Cakoni, F., D. Colton, and D. Gintides: *The interior transmission eigenvalue problem*, SIAM Journal on Mathematical Analysis, 42 (2010), 2912–2921.

[6] Cakoni, F., D. Gintides, and H. Haddar: *The existence of an infinite discrete set of transmission eigenvalues*, SIAM Journal on Mathematical Analysis, 42 (2010), 237-255.

[7] Cakoni, F., and H. Haddar: *Transmission eigenvalues in inverse scattering theory*, in [20], 529–578.

[8] Colton, D., and P. Monk: *The inverse scattering problem for time-harmonic acoustic waves in an inhomogeneous medium*, The Quarterly Journal of Mechanics and Applied Mathematics, 41 (1988), 97–125.

[9] Colton, D., L. Päivärinta and J. Sylvester: *The interior transmission problem*, Inverse Problems and Imaging, 1 (2007), 13–28.

[10] Hickmann, K.: *Interior transmission eigenvalue problem with refractive index having C^2-transition to the background medium*, Applicable Analysis, 91 (2012), 1675–1690.

[11] Hörmander, L.: *The Analysis of Linear Partial Differential Operators II. Differential Operators with Constant Coefficients*, Classics in Mathematics, Springer, 2005.

[12] Kato, T.: *Perturbation Theory for Linear Operators*, Classics in Mathematics, Springer, 1995.

[13] Kirsch, A: *The denseness of the far field patterns for the transmission problem*, IMA Journal of Applied Mathematics, 37 (1986), 213–225.

[14] Lakshtanov, E., and B. Vainberg: *Applications of elliptic operator theory to the isotropic interior transmission eigenvalue problem*, preprint at arXiv:1212.6785 (2012).

[15] Päivärinta, L., and J. Sylvester: *Transmission eigenvalues*, SIAM Journal on Mathematical Analysis, 40 (2008), 738–753.

[16] Robbiano, L.: *Spectral analysis on interior transmission eigenvalues*, preprint at arXiv:1302.4851 (2013).

[17] Serov, V.: *Transmission eigenvalues for non-regular cases*, Communications in Mathematical Analysis, 14 (2013), 129–142.

[18] Serov, V., and J. Sylvester: *Transmission eigenvalues for degenerate and singular cases*, Inverse Problems, 28 (2012), 1–8.

[19] Sylvester, J.: *Discreteness of transmission eigenvalues via upper triangular compact operators*, SIAM Journal on Mathematical Analysis, 44 (2012), 341–354.

[20] Uhlmann, G. (editor): *Inverse Problems and Applications: Inside Out II*, MSRI Publications 60, Cambridge University Press, 2013.

II. Rellich Type Theorems for Unbounded Domains

Esa V. Vesalainen

Abstract

We give several generalizations of Rellich's classical uniqueness theorem to unbounded domains. We give a natural half-space generalization for super-exponentially decaying inhomogeneities using real variable techniques. We also prove under super-exponential decay a discrete generalization where the inhomogeneity only needs to vanish in a suitable cone.

The more traditional complex variable techniques are used to prove the half-space result again, but with less exponential decay, and a variant with polynomial decay, but with supports exponentially thin at infinity. As an application, we prove the discreteness of non-scattering energies for non-compactly supported potentials with suitable asymptotic behaviours and supports.

1 Introduction

1.1 Scattering theory

Our objects of study arise from scattering theory. More precisely, time independent scattering theory for short-range potentials, which models e.g. two-body quantum scattering, acoustic scattering, and some classical electromagnetic scattering situations (for a general reference, see e.g. [10]). Here one is concerned with the situation where, at a fixed energy or wavenumber $\lambda \in \mathbb{R}_+$, an incoming wave w, which is a solution to the free equation

$$(-\Delta - \lambda)\, w = 0,$$

is scattered by some perturbation of the flat homogeneous background. Here this perturbation will be modeled by a real-valued function V in $\mathbb{R}^n$ having enough decay at infinity. The total wave v, which models the "actual" wave, then solves the perturbed equation

$$(-\Delta + V - \lambda)\, v = 0.$$

For acoustic and electromagnetic scattering, one writes λV instead of V. Of course, the two waves v and w must be linked together and the connection is given by the Sommerfeld radiation condition. The upshot will be that the difference u of v and w, the so-called scattered wave, will have an asymptotic expansion of the shape

$$u(x) = v(x) - w(x) = A\left(\frac{x}{|x|}\right) \frac{e^{i\sqrt{\lambda}|x|}}{|x|^{(n-1)/2}} + \text{error},$$

where A depends on λ and w, and where the error term decays more rapidly than the main term. The point here is that in the main term the dependences on the radial and angular variables are neatly separated, and in practical applications one usually measures the scattering amplitude or far-field pattern A, or its absolute value $|A|$.

1.2 Non-scattering energies

It is a natural question whether we can have $A \equiv 0$ for some $w \not\equiv 0$? This would mean that the main term of the scattered wave vanishes at infinity, meaning that the perturbation, for the special incident wave in question, is not seen far away. Values of $\lambda \in \mathbb{R}_+$ for which such an incident wave w exists, are called non-scattering energies (or appropriately, wavenumbers) of V. In order to avoid the discussion of function spaces here, the precise definition is given in Section 2 below.

Results on the existence of non-scattering energies are scarce. Essentially only two general results are known: For compactly supported radial potentials the set of non-scattering energies is an infinite discrete set accumulating at infinity [11], and for compactly supported potentials with suitable corners, Blåsten, Päivärinta and Sylvester recently proved that the set of non-scattering energies is empty [5].

We would like to mention the related topic of transparent potentials: there one considers (at a fixed energy) potentials for which A vanishes for all w. The knowledge of transparent potentials is more extensive. In particular, several constructions of such radial potentials have been given, see e.g. the works of Regge [32], Newton [28], Sabatier [37], Grinevich and Manakov [13], and Grinevich and Novikov [14].

1.3 Rellich type theorems

In practice, discreteness of the set of non-scattering energies tends to be a more attainable goal. The first key step towards that goal (for compactly supported V) is supplied by Rellich's classical uniqueness theorem which is the following:

Theorem 1. *Let $u \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ solve the equation $(-\Delta - \lambda)\, u = f$, where $\lambda \in \mathbb{R}_+$ and $f \in L^2(\mathbb{R}^n)$ is compactly supported, and assume that*

$$\frac{1}{R} \int\limits_{B(0,R)} |u(x)|^2 \, \mathrm{d}x \longrightarrow 0,$$

as $R \longrightarrow \infty$. Then u also is compactly supported.

This was first proved (though with a bit different decay condition) independently by Rellich [33] and Vekua [45] in 1943. Of the succeeding work, which includes generalizations of this result to more general constant coefficient differential operators, we would like to mention the work of Trèves [43], Littman [23, 24, 25], Murata [27] and Hörmander [17]. Section 8 of [16] is also interesting.

We also mention that a theorem analogous to Theorem 1 also exists for the discrete Laplacian (also to be defined more precisely in Section 2):

Theorem 2. *Let $u\colon \mathbb{Z}^n \longrightarrow \mathbb{C}$ be a solution to $(-\Delta_{\mathrm{disc}} - \lambda)\, u = f$, where*

$$\frac{1}{R} \sum_{\xi \in \mathbb{Z}^n, |\xi| \leqslant R} |u(\xi)|^2 \longrightarrow 0,$$

as $R \longrightarrow \infty$, and $f \in \ell^2(\mathbb{Z}^n)$ is non-zero only at finitely many points of $\mathbb{Z}^n$, and $\lambda \in \,]0, n[$. Then u also is non-zero only at finitely many points of $\mathbb{Z}^n$.

This theorem was proved recently by Isozaki and Morioka [19]. A less general version of the result was implicit in the work of Shaban and Vainberg [40].

1.4 Transmission eigenvalues

Assume that V is compactly supported. The equations for v and w imply that the scattered wave u solves the equation

$$(-\Delta - \lambda)\, u = -Vv.$$

If furthermore $A \equiv 0$, then u will satisfy the decay condition in Theorem 1, and so $u = v - w$ will vanish outside a compact set. If the support of V is essentially some suitable open domain Ω, the unique continuation principle for the free Helmholtz equation allows us to conclude that actually

$$\begin{cases} (-\Delta + V - \lambda)\, v = 0 & \text{in } \Omega, \\ (-\Delta - \lambda)\, w = 0 & \text{in } \Omega, \\ v - w \in H_0^2(\Omega). \end{cases}$$

This system, called the interior transmission problem, is a non-self-adjoint eigenvalue problem for λ, and the values of λ, for which this system has non-trivial L^2-solutions, are called (interior) transmission eigenvalues.

The non-scattering energies and transmission eigenvalues first appeared in the papers of Colton and Monk [11] and Kirsch [21]. In [9] Colton, Kirsch and Päivärinta proved the discreteness of transmission eigenvalues (and non-scattering energies) for potentials that may even be mildly degenerate. The early papers on the topic also considered, among other things, radial potentials; for more on this, we refer to the article of Colton, Päivärinta and Sylvester [12].

In recent years, there has been a surge of interest in the topic starting with the general existence results of Päivärinta and Sylvester [30], who established existence of transmission eigenvalues for a large class of potentials, and Cakoni, Gintides and Haddar [6], who established for acoustic scattering, that actually the set of transmission eigenvalues must be infinite.

For potentials more general than the radial ones, a very common approach to proving discreteness and other properties has been via quadratic forms: the scattered wave solves the fourth-order equation

$$(-\Delta + V - \lambda) \frac{1}{V} (-\Delta - \lambda) u = 0,$$

and this can be handled nicely with quadratic forms (or with variational formulations) and analytic perturbation theory.

Recently, other approaches, not involving the fourth-order equation, to proving discreteness and many other results have been introduced by Sylvester [41], Robbiano [34], and Lakshtanov and Vainberg [22].

For more information and a wealth of references on transmission eigenvalues, we recommend the survey of Cakoni and Haddar [7] and their editorial [8].

1.5 What we do and why?

Most of the work on non-scattering energies and transmission eigenvalues deals with compactly supported potentials V. However, the basic short-range scattering theory only requires V to have enough decay at infinity, essentially something like $V(x) \ll |x|^{-1-\varepsilon}$. Thus, it makes perfect sense to study non-scattering energies for non-compactly supported potentials.

In [46], we studied an analogue of the transmission eigenvalue problem for certain unbounded domains establishing discreteness under the assumptions that $V(x) \asymp |x|^{-\alpha}$ for some $\alpha \in]3, \infty[$, and that the underlying domain Ω has the property that the embedding $H_0^2(\Omega) \longrightarrow L^2(\Omega)$ is compact.

Even though the formulation of the transmission problem was kept compatible with scattering theory, the lack of a Rellich type theorem for unbounded domains did not allow any conclusions about the discreteness of non-scattering energies for the corresponding potentials. Here we will present several such Rellich type theorems, two of which will give discreteness of non-scattering energies.

First, we prove a fairly general Rellich type theorem where instead of a compactly supported inhomogeneity f, we consider f that is superexponentially decaying and vanishes in a half-space. The conclusion will then be that the solution u also vanishes in the same half-space. This is a fairly satisfying generalization. Also, it contains the classical Rellich lemma as a simple corollary. We give a new kind of proof for this result based on real variable techniques, first deriving a Carleman estimate weighted exponentially in one direction from an estimate of Sylvester and Uhlmann [42] and then arguing immediately from it.

However, we have so far been unable to apply the quadratic form approach to superexponentially decaying potentials, and so we would like to have Rellich type theorems which allow less decay. We shall give two results of this kind: the first is for exponentially decaying inhomogeneities, the second is essentially for polynomially decaying potentials but for domains that are not only contained in a half-space but also grow exponentially thin at infinity. These results are proved with a more traditional complex variables argument [43, 23, 24, 25, 17].

The Rellich type theorem for polynomially decaying inhomogeneities can be immediately combined with the results of [46] to give discreteness of non-scattering energies for a class of polynomially decaying potentials. Obtaining a discreteness result for a class of exponentially decaying potentials will require some minor adjustments to the arguments of [46] which are presented in Section 5. Our manner of using quadratic forms to establish discreteness is a close relative of the application of quadratic forms to degenerate and singular potentials in the works of Colton, Kirsch and Päivärinta [9], Serov and Sylvester[39], Serov [38], and Hickmann [15].

Finally, as an interesting aside, and to provide a point of comparison, we present a generalization of the discrete Rellich type theorem of Isozaki and Morioka. It turns out that for superexponentially decaying potentials, one gets a much stronger result than in the continuous case: we not only can consider vanishing in half-spaces but vanishing in suitable cones. The proof depends heavily on the arguments in [19] which are first used to show that the solution must be superexponentially decaying. After this, the Rellich type conclusion follows from a repeated application of the definition of the discrete Laplacian.

1.6 On notation

We shall use the standard asymptotic notation. If f and g are complex functions on some set A, then $f \ll g$ means that $|f(x)| \leqslant C\,|g(x)|$ for all $x \in A$ for some positive real constant C, referred to as the implicit constant. The notation $f \asymp g$ means that both $f \ll g$ and $g \ll f$. When C may depend on some parameters α, β, …, we write $f \ll_{\alpha,\beta,\ldots} g$, except that all the implicit constants are allowed to depend on the dimension $n \in \{2, 3, \ldots\}$ of the ambient Euclidean space $\mathbb{R}^n$.

For a vector $x \in \mathbb{R}^n$, we define $\langle x \rangle = \sqrt{1 + |x|^2}$ and $x' = \langle x_1, \ldots, x_{n-1} \rangle$. The letters e_1, e_2, …, e_n denote the standard basis of $\mathbb{R}^n$:

$$\begin{aligned} e_1 &= \langle 1, 0, 0, \ldots, 0, 0 \rangle\,, \\ e_2 &= \langle 0, 1, 0, \ldots, 0, 0 \rangle\,, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ e_n &= \langle 0, 0, 0, \ldots, 0, 1 \rangle\,. \end{aligned}$$

For a complex vector $z \in \mathbb{C}^n$, we denote the real and imaginary parts of z by

$$\Re z = \langle \Re z_1, \ldots, \Re z_n \rangle \quad \text{and} \quad \Im z = \langle \Im z_1, \ldots, \Im z_n \rangle\,.$$

For $R \in \mathbb{R}_+$, we write $B(0, R)$ for the open ball of vectors $\xi \in \mathbb{R}^n$ with $|\xi| < R$. If we want to emphasize the dimension of the ambient Euclidean space, we write $B^n(0, R)$.

We shall use the shorthand $\nabla \otimes \nabla$ to simplify expressions of Sobolev norms in the obvious way. For example, the usual H^2-Sobolev norm would be given by the expression

$$\sqrt{\|u\|^2 + \|\nabla u\|^2 + \|\nabla \otimes \nabla u\|^2}.$$

We use $\mathbb{T}^n$ to denote the n-dimensional torus $\mathbb{R}^n/\mathbb{Z}^n$. The corresponding complex torus $\mathbb{T}^n_{\mathbb{C}}$ means $\mathbb{C}^n/\mathbb{Z}^n$. Instead of analytic functions in $\mathbb{T}^n_{\mathbb{C}}$ one can simply think of entire functions in $\mathbb{C}^n$ which are 1-periodic with respect to each complex variable.

For a function $f \in L^2(\mathbb{R}^n)$, $\widehat{f}$ denotes the usual Fourier transform normalized as follows: for a Schwartz test function f and $\xi \in \mathbb{R}^n$,

$$\widehat{f}(\xi) = \int\limits_{\mathbb{R}^n} f(x)\, e(-x \cdot \xi)\, \mathrm{d}x,$$

where $e(\cdot)$ stands for $e^{2\pi i \cdot}$. We will also denote by $F'f$ the Fourier transform of f in the first $n-1$ variables, and the corresponding inverse transform by F'^{-1}.

Similarly, we will denote by F_n the Fourier transform in the nth variable, and by F_n^{-1} the corresponding inverse transform.

In the same vein, given a vector $f \in \ell^2(\mathbb{Z}^n)$, $\check{f}$ denotes the Fourier series connected to f: for $x \in \mathbb{T}^n$,

$$\check{f}(x) = \sum_{\xi \in \mathbb{Z}^n} f(x)\, e(x \cdot \xi),$$

and the convergence is in the L^2-sense or pointwise, whichever is more appropriate.

Finally, the usual L^2-inner product is denoted by $\langle \cdot | \cdot \rangle$: for square-integrable complex functions f and g in some domain Ω, we define

$$\langle f|g \rangle = \int_\Omega \overline{f}\, g.$$

2 The results

2.1 Rellich type theorems for unbounded domains

Before stating the main results, we would like to define the function spaces from which solutions v, w and u are actually taken. The solutions v and w should be taken from the Agmon–Hörmander space B^*, which consists of those $L^2_{\mathrm{loc}}(\mathbb{R}^n)$-functions u for which

$$\|u\|^2_{B^*} = \sup_{R>1} \frac{1}{R} \int_{B(0,R)} |u(x)|^2 \,\mathrm{d}x < \infty.$$

Actually, it will then turn out that also the first- and second-order partial derivatives of v and w will also belong to B^*, a fact which we will denote by $v, w \in B^*_2$.

When $A \equiv 0$, the scattered wave u will satisfy

$$\frac{1}{R} \int_{B(0,R)} |u(x)|^2 \,\mathrm{d}x \longrightarrow 0$$

as $R \longrightarrow \infty$. The space of such $L^2_{\mathrm{loc}}(\mathbb{R}^n)$-functions is denoted by $\mathring{B}^*$. From Theorem 14.3.6 of [18], it will follow that the first- and second-order partial derivatives of u will also belong to $\mathring{B}^*$, and we will write $u \in \mathring{B}^*_2$. In view of this, the growth condition for u in Theorem 1 could be replaced by $u \in \mathring{B}^*_2$, and this is what we shall do in the theorems below.

The Agmon–Hörmander spaces were introduced in the works of Agmon and Hörmander [4], and independently by Murata [26], in the study of asymptotics of solutions to constant coefficient partial differential equations. Despite their appearance, these spaces turn out to be fairly natural. In particular, the mapping from $g \in L^2(S^{n-1})$ to the Herglotz wave

$$w(x) = \int_{S^{n-1}} g(\vartheta)\, e^{i\sqrt{\lambda}\vartheta\cdot x}\, \mathrm{d}\vartheta$$

will give a bijection from $L^2(S^{n-1})$ into the B^*-solutions of $(-\Delta - \lambda)\, w = 0$, and the respective norms of g and the Herglotz wave w will be comparable.

Our first generalization of Theorem 1 considers vanishing in a half-space instead of the exterior of a ball.

Theorem 3. *Let* $u \in \mathring{B}_2^*$ *solve*

$$(-\Delta - \lambda)\, u = f,$$

where $\lambda \in \mathbb{R}_+$ *and* $f \in e^{-\gamma\langle\cdot\rangle} L^2(\mathbb{R}^n)$ *for all* $\gamma \in \mathbb{R}_+$*, and suppose that* f *vanishes in the lower half-space* $\mathbb{R}^{n-1} \times \mathbb{R}_-$*. Then also* u *vanishes in* $\mathbb{R}^{n-1} \times \mathbb{R}_-$*.*

With complex variable techniques, we may allow less exponential decay:

Theorem 4. *Let* $u \in \mathring{B}_2^*$ *solve*

$$(-\Delta - \lambda)\, u = f,$$

where $\lambda \in \mathbb{R}_+$ *and* $f \in e^{-\gamma_0\langle\cdot\rangle} L^2(\mathbb{R}^n)$ *for some* $\gamma_0 \in \mathbb{R}_+$*, and suppose that* f *vanishes in* $\mathbb{R}^{n-1} \times \mathbb{R}_-$*. Then also* u *vanishes in* $\mathbb{R}^{n-1} \times \mathbb{R}_-$*.*

Finally, to allow only polynomially decaying f we assume essentially that the support of the regions where f is not exponentially decaying is exponentially thin at infinity:

Theorem 5. *Let* $u \in \mathring{B}_2^*$ *solve*

$$(-\Delta - \lambda)\, u = f,$$

where $\lambda \in \mathbb{R}_+$ *and the inhomogeneity* f *both vanishes in* $\mathbb{R}^{n-1} \times \mathbb{R}_-$ *and satisfies*

$$\sup_{\substack{\zeta' \in \mathbb{C}^{n-1} \\ |\Im\zeta'| < \gamma_0}} \int_{\mathbb{R}^n} \left| \langle x\rangle^2\, f(x)\, e(x' \cdot \zeta') \right| \mathrm{d}x < \infty$$

for some $\gamma_0 \in \mathbb{R}_+$*. Then* u *vanishes in the lower half-space* $\mathbb{R}^{n-1} \times \mathbb{R}_-$*.*

2.2 Applications to discreteness of non-scattering energies

The next two theorems will concern discreteness of non-scattering energies in a somewhat special situations. The quadratic form techniques in Section 5 depend on the potential V being essentially supported in a domain Ω for which the embedding $H_0^2(\Omega) \longrightarrow L^2(\Omega)$ is compact.

For a discussion of such compact embeddings for unbounded domains, we refer to Chapter 6 of [2] or to the original article [1]. The conditions get slightly more involved in higher dimensions, but in two- and three-dimensional cases, the embedding $H_0^2(\Omega) \longrightarrow L^2(\Omega)$ is compact if and only if the domain Ω does not contain an infinite sequence of pairwise disjoint congruent balls; cf. remarks 6.17.3, 6.9 and 6.11 in [2].

Combining Theorem 5 with the results in [46] easily gives the following discreteness result for non-scattering energies:

Theorem 6. *Let $V \in L^\infty(\mathbb{R}^n)$ take only nonnegative real values, and let $\Omega \subseteq \mathbb{R}^{n-1} \times \mathbb{R}_+$ be a non-empty open set for which the Sobolev embedding $H_0^2(\Omega) \longrightarrow L^2(\Omega)$ is compact. Assume the following:*

I. *$V(\cdot) \asymp \langle\cdot\rangle^{-\alpha}$ in Ω for some $\alpha \in \,]3, \infty[$, and V vanishes in $\mathbb{R}^n \setminus \Omega$.*

II. *The complement of Ω in $\mathbb{R}^n$ has a connected interior and is the closure of the interior.*

III. *The integrals*
$$\int_\Omega \left| \langle x \rangle^{2+1/2+\varepsilon-\alpha} \; e(x \cdot \zeta) \right|^2 \mathrm{d}x$$
are uniformly bounded for all $\zeta \in \mathbb{C}^n$ with $|\Im\zeta| < \gamma_0$ for some $\gamma_0, \varepsilon \in \mathbb{R}_+$.

Then the set of non-scattering energies for V is a discrete subset of $[0, \infty[$, and each of them is of finite multiplicity.

Here multiplicity is defined to be the dimension of the vector space of pairs of solutions $\langle v, w \rangle$ appearing in the definition of non-scattering energies. The point of the condition III is that, combined with the Cauchy–Schwarz inequality, and the definition of B^*, it guarantees that Vv satisfies the sup-condition of Theorem 5 for any $v \in B^*$. Since the exponential factor in the condition III grows exponentially fast in some directions when $\Im\zeta \neq 0$, the condition essentially says that Ω must be exponentially thin at infinity.

Analogously to the Rellich type theorems, if the condition of polynomial decay is replaced by exponential decay, then we can relax the conditions:

Theorem 7. *Let $V \in L^\infty(\mathbb{R}^n)$ take only nonnegative real values, and let $\Omega \subseteq \mathbb{R}^{n-1} \times \mathbb{R}_+$ be a non-empty open set for which the Sobolev embedding $H^2_0(\Omega) \longrightarrow L^2(\Omega)$ is compact. Assume the following:*

I. *$V(\cdot) \asymp e^{-\gamma_0 \langle \cdot \rangle}$ in Ω for some $\gamma_0 \in \mathbb{R}_+$ with $\gamma_0 \ll_n 1$, and V vanishes in $\mathbb{R}^n \setminus \Omega$.*

II. *The complement of Ω in $\mathbb{R}^n$ has a connected interior and is the closure of the interior.*

Then the set of non-scattering energies for V is a discrete subset of $[0, \infty[$ and each of them is of finite multiplicity.

The proof will be similar to the arguments in [46]. We will outline the relevant modifications in Section 5.

2.3 A discrete Rellich type theorem for unbounded domains

Let $u \colon \mathbb{Z}^n \longrightarrow \mathbb{C}$ be a function on the square lattice $\mathbb{Z}^n$. Then we define the discrete Laplacian of u to be the function $-\Delta_{\mathrm{disc}} u \colon \mathbb{Z}^n \longrightarrow \mathbb{C}$ given by the formula

$$\big(-\Delta_{\mathrm{disc}} u\big)(\xi) = \frac{n}{2} u(\xi) - \frac{1}{4} \sum_{\ell=1}^{n} \big(u(\xi + e_\ell) + u(\xi - e_\ell)\big).$$

The spectrum of $-\Delta_{\mathrm{disc}}$ is $[0, n]$ and absolutely continuous. For more information about the discrete setting, we refer to the presentation of Isozaki and Morioka [19] and the references given there.

Our generalization of Theorem 2 will concern vanishing, not in a half-space, but in a cone-like domain.

Theorem 8. *Let C be the set of those $\xi \in \mathbb{Z}^n$ for which*

$$|\xi_1| + |\xi_2| + \ldots + |\xi_{n-1}| \leqslant \xi_n.$$

Also, let $u \colon \mathbb{Z}^n \longrightarrow \mathbb{C}$ be such that

$$\frac{1}{R} \sum_{|\xi| \leqslant R} \big|u(\xi)\big|^2 \longrightarrow 0$$

as $R \longrightarrow \infty$, and let $f \in \ell^2(\mathbb{Z}^n)$ be such that

$$e^{\gamma \langle \cdot \rangle} f \in \ell^2(\mathbb{Z}^n)$$

for all $\gamma \in \mathbb{R}_+$*, and assume that* $f(\xi) = 0$ *for all* $\xi \in C$*. Finally, let* $\lambda \in \left]0, n\right[$*, and assume that*

$$(-\Delta_{\mathrm{disc}} - \lambda)\, u = f$$

in $\mathbb{Z}^n$*. Then also* $u(\xi) = 0$ *for all* $\xi \in C$*.*

3 Proof of Theorem 3 via Carleman estimates

The following is essentially Lemma 2.5 in [29, p. 1780], which may be reformulated in the following way in view of Theorem 14.3.6 in [18].

Theorem 9. *Let* $\gamma_0 \in \mathbb{R}_+$*, and let* f *be a function such that* $e^{\gamma \langle\cdot\rangle} f \in L^2(\mathbb{R}^n)$ *for each* $\gamma \in \left]0, \gamma_0\right[$*. If* $u \in \mathring{B}_2^*$ *solves the equation*

$$(-\Delta - \lambda)\, u = f,$$

then $e^{\gamma \langle\cdot\rangle} u \in H^2(\mathbb{R}^n)$ *for each* $\gamma \in \left]0, \gamma_0\right[$*.*

The following result is Proposition 2.1 in [42]. Even though the statement there has $\lambda = 0$, the same proof also works for $\lambda \geqslant 0$.

Theorem 10. *Let* $\lambda \in \mathbb{R}_+$*,* $\delta \in \left]-1, 0\right[$*, and let* $\rho \in \mathbb{C}^n$ *with* $\rho \cdot \rho = \lambda$ *and* $|\Im \rho| \geqslant 1$*. Then for any* $f \in \langle\cdot\rangle^{-1-\delta} L^2(\mathbb{R}^n)$ *the equation*

$$(-\Delta - 2i\rho \cdot \nabla)\, v = f$$

has a unique solution $v \in \langle\cdot\rangle^{-\delta} L^2(\mathbb{R}^n)$ *satisfying the estimate*

$$\left\| \langle\cdot\rangle^{\delta} v \right\|_{L^2(\mathbb{R}^n)} \ll_{\lambda,\delta} \frac{1}{|\rho|} \left\| \langle\cdot\rangle^{1+\delta} f \right\|_{L^2(\mathbb{R}^n)}.$$

This result can be turned into a Carleman estimate weighted exponentially in one coordinate direction:

Corollary 11. *Let* $u \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$*,* $\delta \in \left]-1, 0\right[$*,* $\lambda \in \mathbb{R}_+$*, and let* $\tau \in \mathbb{R}_+$ *with* $\tau \gg_{\lambda,\delta} 1$*. Then*

$$\left\| e^{-\tau x_n} \langle\cdot\rangle^{\delta} u \right\|_{L^2(\mathbb{R}^n)} \ll_{\lambda,\delta} \frac{1}{\tau} \left\| e^{-\tau x_n} \langle\cdot\rangle^{\delta+1} (-\Delta - \lambda)\, u \right\|_{L^2(\mathbb{R}^n)}.$$

Of course, this also holds for u in the closure of $C_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$ in the norm that is given by the sum of the norms appearing in the estimate.

Proof. We shall apply Theorem 10 with $\rho = \alpha - i\tau e_n$, where $e_n = \langle 0, 0, \dots, 0, 1\rangle$ and $\alpha \in \mathbb{R}^n$ is such that $\alpha \cdot e_n = 0$ and $|\alpha|^2 = \tau^2 + \lambda$, as well as with $v = e^{-i\rho\cdot x}u$. With these choices we have

$$\rho\cdot\rho = \lambda, \quad |\rho| \asymp_\lambda \tau, \quad \text{and} \quad \left|e^{-i\rho\cdot x}\right| = e^{-\tau x_n},$$

as well as

$$(-\Delta - 2i\rho\cdot\nabla)\, v = e^{-i\rho\cdot x}\,(-\Delta - \lambda)\, u.$$

Now Theorem 10 clearly says that

$$\left\|e^{-\tau x_n}\,\langle\cdot\rangle^{\delta}\, u\right\|_{L^2(\mathbb{R}^n)} \ll_{\lambda,\delta} \frac{1}{\tau}\left\|e^{-\tau x_n}\,\langle\cdot\rangle^{\delta+1}\,(-\Delta-\lambda)\, u\right\|_{L^2(\mathbb{R}^n)}.$$

Proof of Theorem 3. Let us first observe that by Theorem 9 we must have $\partial^\alpha u \in e^{-\gamma\langle\cdot\rangle}L^2(\mathbb{R}^n)$ for each multi-index α with $|\alpha| \leqslant 2$. In particular, we also have $e^{-\tau x_n}\,\langle\cdot\rangle^{\delta+1}\,\partial^\alpha u \in L^2(\mathbb{R}^n)$ for all $\tau \in \mathbb{R}$, any fixed $\delta \in\,]-1, 0[$ and each multi-index α with $|\alpha| \leqslant 2$.

Our goal will be to prove that u vanishes in $\mathbb{R}^{n-1}\times\,]-\infty, -2[$. Then u also vanishes in $\mathbb{R}^{n-1}\times\mathbb{R}_-$ by the unique continuation property of solutions to the free Helmholtz equation.

We want to focus on the behaviour of u in the lower half-space and so we will pick a cut-off function $\chi \in C^\infty(\mathbb{R}^n)$ which only depends on x_n, vanishes for $x_n > -1$, and is identically equal to 1 when $x_n < -2$.

Now Corollary 11 tells us that for large $\tau \in \mathbb{R}_+$,

$$\begin{aligned}
&e^{2\tau}\left\|\langle\cdot\rangle^{\delta}\, u\right\|_{L^2(\mathbb{R}^{n-1}\times]-\infty,-2[)}\\
&\quad\ll \left\|e^{-\tau x_n}\,\langle\cdot\rangle^{\delta}\,\chi u\right\|_{L^2(\mathbb{R}^n)}\\
&\quad\ll_{\lambda,\delta} \frac{1}{\tau}\left\|e^{-\tau x_n}\,\langle\cdot\rangle^{\delta+1}\,(-\Delta-\lambda)\,(\chi u)\right\|_{L^2(\mathbb{R}^n)}\\
&\quad= \frac{1}{\tau}\left\|e^{-\tau x_n}\,\langle\cdot\rangle^{\delta+1}\left(2\,\frac{\partial\chi}{\partial x_n}\cdot\frac{\partial u}{\partial x_n} + \frac{\partial^2\chi}{\partial x_n^2}\,u\right)\right\|_{L^2(\mathbb{R}^{n-1}\times]-2,-1[)}\\
&\quad\ll \frac{1}{\tau}e^{2\tau}\left\|\langle\cdot\rangle^{\delta+1}\left(2\,\frac{\partial\chi}{\partial x_n}\cdot\frac{\partial u}{\partial x_n} + \frac{\partial^2\chi}{\partial x_n^2}\,u\right)\right\|_{L^2(\mathbb{R}^{n-1}\times]-2,-1[)}
\end{aligned}$$

and the result follows by dividing by $e^{2\tau}$ and letting $\tau \longrightarrow \infty$.

4 Proofs of Theorems 4 and 5

4.1 Differentiation under integral signs

The following lemmas have been adapted from the first chapter of Wong's textbook [47]. The reason for them is that we apply analytic continuations for Fourier transforms, and to obtain these extensions with only polynomial decay, we will differentiate under the integral sign in order to check that the Cauchy–Riemann equations hold.

Lemma 12. *Let D and Ω be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and let $f\colon D \times \Omega \longrightarrow \mathbb{C}$ be measurable. Suppose that*

I. $f(x,\cdot) \in L^1(\Omega)$ *for each* $x \in D$*;*

II. $f(\cdot,y) \in C^2(D)$ *for almost every* $y \in \Omega$*; and that*

III. *for multi-indices* α *with* $|\alpha| \leqslant 2$*, we have*

$$\sup_{x\in D} \int_\Omega |\partial_x^\alpha f(x,y)|\,\mathrm{d}y < \infty.$$

Then, for each $\ell \in \{1,2,\dots,n\}$*, the integrals*

$$\int_\Omega \big(\partial_{x_\ell} f\big)(x,y)\,\mathrm{d}y \quad \text{and} \quad \int_\Omega \big|\big(\partial_{x_\ell} f\big)(x,y)\big|\,\mathrm{d}y$$

are uniformly continuous functions of $x \in D$*.*

Proof. For each $k \in \{1,2,\dots,n\}$ and small $h \in \mathbb{R}$, we may apply the mean value theorem to estimate

$$\begin{aligned}
&\int_\Omega \big|\big(\partial_{x_\ell} f\big)(x_1,\dots,x_k+h,\dots,x_n,y) - \big(\partial_{x_\ell} f\big)(x_1,\dots,x_k,\dots,x_n,y)\big|\,\mathrm{d}y \\
&= |h| \int_\Omega \big|\big(\partial_{x_k}\partial_{x_\ell} f\big)(x_1,\dots,\xi(x,y;h),\dots,x_n,y)\big|\,\mathrm{d}y \\
&\leqslant |h| \sup_{x\in D} \int_\Omega \big|\big(\partial_{x_k}\partial_{x_\ell} f\big)(x,y)\big|\,\mathrm{d}y.
\end{aligned}$$

Here $\xi(\dots)$ has the obvious meaning.

Theorem 13. *Let D and Ω be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and let $f\colon D \times \Omega \longrightarrow \mathbb{C}$ be measurable. Suppose that*

I. *$f(x, \cdot) \in L^1(\Omega)$ for each $x \in D$;*

II. *$f(\cdot, y) \in C^2(D)$ for almost every $y \in \Omega$; and that*

III. *for all multi-indices α with $|\alpha| \leqslant 2$, we have*

$$\sup_{x \in D} \int_{\Omega} \bigl|\bigl(\partial_x^\alpha f\bigr)(x, y)\bigr| \,\mathrm{d}y < \infty.$$

Then

$$\int_{\Omega} f(\cdot, y) \,\mathrm{d}y \in C^1(D)$$

and

$$\partial_{x_\ell} \int_{\Omega} f(x, y) \,\mathrm{d}y = \int_{\Omega} \bigl(\partial_{x_\ell} f\bigr)(x, y) \,\mathrm{d}y$$

in D for each $\ell \in \{1, 2, \ldots, n\}$.

Proof. By the fundamental theorem of analysis, the previous lemma and Fubini's theorem,

$$\begin{aligned}
&\partial_{x_\ell} \int_{\Omega} f(x, y) \,\mathrm{d}y \\
&= \lim_{h \longrightarrow 0} \frac{1}{h} \int_{\Omega} \bigl(f(x_1, \ldots, x_\ell + h, \ldots, x_n, y) - f(x_1, \ldots, x_\ell, \ldots, x_n, y)\bigr) \,\mathrm{d}y \\
&= \lim_{h \longrightarrow 0} \frac{1}{h} \int_{\Omega} \int_{x_\ell}^{x_\ell + h} \bigl(\partial_{x_\ell} f\bigr)(x_1, \ldots, x_{\ell-1}, s, x_{\ell+1}, \ldots, x_n, y) \,\mathrm{d}s \,\mathrm{d}y \\
&= \lim_{h \longrightarrow 0} \frac{1}{h} \int_{x_\ell}^{x_\ell + h} \int_{\Omega} \bigl(\partial_{x_\ell} f\bigr)(x_1, \ldots, x_{\ell-1}, s, x_{\ell+1}, \ldots, x_n, y) \,\mathrm{d}y \,\mathrm{d}s \\
&= \int_{\Omega} \bigl(\partial_{x_\ell} f\bigr)(x, y) \,\mathrm{d}y.
\end{aligned}$$

4.2 Some Paley–Wiener theorems

The following classical result is e.g. Theorem 19.2 in Rudin's textbook [35].

Theorem 14. *A function $f \in L^2(\mathbb{R})$ is supported in $[0,\infty[$* ***if and only if*** *its Fourier transform $\widehat{f}$ extends to an analytic function in the lower half-plane $\{z \in \mathbb{C} | \Im z < 0\}$ and this continuation satisfies*

$$\sup_{\eta \in \mathbb{R}_-} \left\| \widehat{f}(\cdot + i\eta) \right\|_{L^2(\mathbb{R})} < \infty.$$

The following Paley–Wiener theorem is e.g. Theorem XI.13 in [31, p. 18].

Theorem 15. *Let $\gamma_0 \in \mathbb{R}_+$, and let $f \in L^2(\mathbb{R}^n)$. Then $e^{\gamma\langle\cdot\rangle} f \in L^2(\mathbb{R}^n)$ for each $\gamma \in]0,\gamma_0[$* ***if and only if*** *the Fourier transform $\widehat{f} \in L^2(\mathbb{R}^n)$ extends analytically to the set $\{\zeta \in \mathbb{C}^n | |\Im\zeta| < \gamma_0\}$ so that, for each $\eta \in \mathbb{R}^n$ with $|\eta| < \gamma_0$, we have $\widehat{f}(\cdot + i\eta) \in L^2(\mathbb{R}^n)$, and that for each $\gamma \in]0,\gamma_0[$,*

$$\sup_{\substack{\eta \in \mathbb{R}^n \\ |\eta| \leqslant \gamma}} \left\| \widehat{f}(\cdot + i\eta) \right\|_{L^2(\mathbb{R}^n)} < \infty.$$

4.3 Division by the symbol on the Fourier side

For the rest of this section we will simplify our notation by writing p for the symbol polynomial $4\pi^2\left(z_1^2 + z_2^2 + \ldots + z_n^2\right)$. We shall also consider the level-set manifolds

$$M_\lambda^{\mathbb{R}} = \{\xi \in \mathbb{R}^n \,|\, p(\xi) = \lambda\},$$

and

$$M_\lambda^{\mathbb{C}} = \{\zeta \in \mathbb{C}^n \,|\, p(\zeta) = \lambda\},$$

where $\lambda \in \mathbb{R}_+$, as usual.

Lemma 16. *Let $D \subseteq \mathbb{C}^n$ be an open set such that $M_\lambda^{\mathbb{R}} \subseteq D$ and $M_\lambda^{\mathbb{C}} \cap D$ is connected. If $f\colon D \longrightarrow \mathbb{C}$ is analytic and vanishes on the real sphere $M_\lambda^{\mathbb{R}}$, then f also vanishes in the intersection $M_\lambda^{\mathbb{C}} \cap D$ and the expression $f/(p-\lambda)$ gives rise to an analytic function in D.*

The proof is modelled after a portion of the proof of Lemma 2.5 in [29].

Proof. Let $\xi \in M_\lambda^{\mathbb{R}}$ be arbitrary. At least one coordinate of ξ must be non-zero, say $\xi_n \neq 0$. Thus, we have $\nabla p = 8\pi^2 \xi \neq 0$ at ξ, and by the inverse function theorem, the mapping

$$\varphi = \zeta \longmapsto \langle \zeta', p(\zeta) - \lambda \rangle : W \longrightarrow \mathbb{C}^n$$

is a biholomorphic diffeomorphism between an open connected neighbourhood W of ξ in D and $\varphi[W]$. This particular mapping "straightens out" a portion of the sphere around ξ:

$$\varphi\big[M_\lambda^{\mathbb{C}} \cap W\big] = \big\{\zeta \in \varphi[W] \;\big|\; \zeta_n = 0\big\}.$$

The function $f \circ \varphi^{-1} \colon \varphi\big[W\big] \longrightarrow \mathbb{C}$ vanishes on $\varphi\big[W\big] \cap \big(\mathbb{R}^{n-1} \times \{0\}\big)$. Therefore it also vanishes on $\varphi\big[W\big] \cap \big(\mathbb{C}^{n-1} \times \{0\}\big)$, which in turn implies that

$$f\big|_W = f \circ \varphi^{-1} \circ \varphi$$

vanishes in $M_\lambda^{\mathbb{C}} \cap W$.

Now f vanishes in a neighbourhood of $M_\lambda^{\mathbb{R}}$ in $M_\lambda^{\mathbb{C}} \cap D$ and must vanish everywhere on $M_\lambda^{\mathbb{C}} \cap D$ as an analytic function on a connected analytic manifold.

The second part of the lemma is proved by a similar reasoning. We pick again an arbitrary point ξ, but this time from the complex sphere $M_\lambda^{\mathbb{C}} \cap D$. Again we will have a biholomorphic diffeomorphism φ from some open connected neighbourhood W of ξ in D into $\varphi[W]$, given by the same expression as before. Since $f \circ \varphi^{-1}$ vanishes on $\varphi[W] \cap \big(\mathbb{C}^{n-1} \times \{0\}\big)$, the expression $\big(f \circ \varphi^{-1}\big)(\zeta)/\zeta_n$ defines a function analytic in $\varphi[W]$. Finally, we conclude that

$$\frac{f}{p - \lambda} = \left(\zeta \longmapsto \frac{\big(f \circ \varphi^{-1}\big)(\zeta)}{\zeta_n}\right) \circ \varphi$$

is analytic in W, and since ξ was arbitrary, and since $\frac{f}{p-\lambda}$ is definitely analytic outside of $M_\lambda^{\mathbb{C}}$, we are done.

4.4 Proving Theorems 4 and 5

The following proof works verbatim for both Theorems 4 and 5. The only difference is in the reasons for the analytic continuations of $\widehat{f}$. The proof is modelled after the proof of Lemma 2.5 in [29] and the proof of Theorem 8.3 in [16].

Taking Fourier transforms of both sides of the Helmholtz equation gives

$$(p - \lambda)\, \widehat{u} = \widehat{f},$$

which holds in $\mathbb{R}^n$. A basic result in scattering theory, Theorem 14.3.6 from [18], says that $\widehat{f}\,\big|_{M_\lambda^{\mathbb{R}}} \equiv 0$.

The assumptions on f guarantee that $\widehat{f}$ extends to an analytic function in

$$D = \left\{\zeta \in \mathbb{C}^n \;\middle|\; |\Im\zeta| < \gamma_0\right\},$$

where the constant $\gamma_0 \in \mathbb{R}_+$ is the same as in the assumptions of the theorem. For Theorems 4, this follows immediately from Theorem 15. For Theorem 5 we use Theorem 13 which guarantees that the Fourier transform $\widehat{f}$ can be differentiated under the integral sign and so, from the Cauchy–Riemann equations for the integrand, we get the Cauchy–Riemann equations for $\widehat{f}$.

In any case, combining the above facts with Lemma 16 leads to the conclusion that the expression $\widehat{f}/(p-\lambda)$ gives rise to an analytic function in D. In particular, $\widehat{u}$ has an analytic extension to D.

Next, let us fix a point $\xi' \in B^{n-1}(0, \sqrt{\lambda}/2\pi) \subseteq \mathbb{R}^{n-1}$. For clarity, we write $q(\cdot)$ for $p(\xi', \cdot)$. Then $q - \lambda$ is an entire function of one complex variable, and its only zeroes are simple ones at the points

$$\pm\mu = \pm\frac{1}{2\pi}\sqrt{\lambda - 4\pi^2\,|\xi'|^2}.$$

Since f vanishes in $\mathbb{R}^{n-1} \times \mathbb{R}_-$, the Fourier transform $F'f(\xi', \cdot)$ vanishes in $\mathbb{R}_-$, so that by Theorem 14 and the analytic extension of $\widehat{f}$ to D, $\widehat{f}$ has an analytic extension in the last variable to $\mathbb{R} \times i\,]-\infty, \gamma_0[$, and

$$\int_{-\infty}^{\infty} \left|\widehat{f}(\xi', \xi_n - i\eta)\right|^2 \mathrm{d}\xi_n \ll \int_{-\infty}^{\infty} \left|\widehat{f}(\xi', \xi_n)\right|^2 \mathrm{d}\xi_n < \infty$$

for all $\eta \in \mathbb{R}_+$. Of course, $\widehat{u}(\xi', \cdot)$ has an analytic extension to $\mathbb{R} \times i\,]-\infty, \gamma_0[$ as well.

Since $|q(z) - \lambda|$ is bounded from below, when $z \in \mathbb{C}$ and $\Im z < -1$, we have

$$\int_{-\infty}^{\infty} \left|\widehat{u}(\xi', \xi_n - i\eta)\right|^2 \mathrm{d}\xi_n = \int_{-\infty}^{\infty} \left|\frac{\widehat{f}(\xi', \xi_n - i\eta)}{q(\xi_n - i\eta) - \lambda}\right|^2 \mathrm{d}\xi_n \ll \int_{-\infty}^{\infty} \left|\widehat{f}(\xi', \xi_n)\right|^2 \mathrm{d}\xi_n,$$

whenever $\eta \in [1, \infty[$. In the same vein, the expression

$$\left(\int_{-\infty}^{-2\mu} + \int_{2\mu}^{\infty}\right) \left|\widehat{u}(\xi', \xi_n - i\eta)\right|^2 \mathrm{d}\xi_n$$

is also bounded by a constant independent of η, whenever $\eta \in [0,1[$.

Also, since the function $\widehat{f}(\xi',\cdot)/(q-\lambda)$ is analytic in a neighbourhood of the rectangle $[-2\mu,2\mu] \times i\,[-1,0] \subseteq \mathbb{C}$, it is bounded as well, and so

$$\int_{-2\mu}^{2\mu} \left|\widehat{u}(\xi',\xi_n - i\eta)\right|^2 \mathrm{d}\xi_n$$

is bounded from above by something constant and independent of η, even for $\eta \in [0,1[$.

Now we are able to conclude from Theorem 14 that

$$F'u(\xi',x_n) = F_n^{-1}\widehat{u}(\xi',x_n) = 0$$

for all $\xi' \in B^{n-1}\left(0, \frac{\sqrt{\lambda}}{2\pi}\right)$ and $x_n \in \mathbb{R}_-$. Since $F'u$ is analytic with respect to the first $n-1$ variables, $F'u(\xi',x_n) = 0$ for all $\xi' \in \mathbb{R}^{n-1}$ and all $x_n \in \mathbb{R}_-$. Finally, taking F'^{-1} gives the desired conclusion that u vanishes in $\mathbb{R}^{n-1} \times \mathbb{R}_-$.

5 Proof of Theorem 7

Let λ be a non-scattering energy for V. Then

$$\begin{cases} (-\Delta + V - \lambda)\, v = 0, \\ (-\Delta - \lambda)\, w = 0 \end{cases}$$

in $\mathbb{R}^n$ for some $v, w \in B_2^* \setminus 0$ with $v - w \in \mathring{B}_2^*$. Now consider $u = v - w$, which solves

$$(-\Delta - \lambda)\, u = -Vv.$$

By Theorem 4, the assumptions of Theorem 7 and the unique continuation principle for the Helmholtz equation, the function u vanishes in $\mathbb{R}^n \setminus \Omega$. Furthermore, Theorem 9 says that u belongs to the space $H_0^2(\Omega; e^{\gamma\langle\cdot\rangle})$ with $\gamma = \gamma_0/2$, which is the closure of test functions $u \in C_c^\infty(\Omega)$ with respect to the weighted Sobolev norm

$$\left\|e^{\gamma\langle\cdot\rangle}u\right\| + \left\|e^{\gamma\langle\cdot\rangle}\nabla u\right\| + \left\|e^{\gamma\langle\cdot\rangle}\nabla \otimes \nabla u\right\|,$$

where $\|\cdot\|$ denotes the usual L^2-norm.

We shall use $H_0^2(\Omega; e^{\gamma\langle\cdot\rangle})$ as a quadratic form domain. As the ambient Hilbert space we shall use the space $L^2(\Omega; e^{\gamma\langle\cdot\rangle})$, defined in the obvious way by the weighted norm $\left\|e^{\gamma\langle\cdot\rangle}u\right\|$.

Let us consider the composition of mappings

$$H_0^2(\Omega; e^{\gamma\langle\cdot\rangle}) \longrightarrow H_0^2(\Omega) \longrightarrow L^2(\Omega) \longrightarrow L^2(\Omega; e^{\gamma\langle\cdot\rangle}),$$

where the middle mapping is the compact embedding, and the first and the last mappings are just multiplication and division by $e^{\gamma\langle\cdot\rangle}$, respectively. We easily see that the first and last mappings are bounded, and so $H_0^2(\Omega; e^{\gamma\langle\cdot\rangle})$ embeds compactly into $L^2(\Omega; e^{\gamma\langle\cdot\rangle})$.

We have now reduced the situation to a single fourth-order equation:

Lemma 17. *Under the assumptions of Theorem 7, if $\lambda \in \mathbb{R}_+$ is a non-scattering energy, then there exists a function $u \in H_0^2(\Omega; e^{\gamma\langle\cdot\rangle}) \setminus 0$ solving the fourth-order equation*

$$(-\Delta + V - \lambda)\,\frac{1}{V}\,(-\Delta - \lambda)\,u = 0 \tag{1}$$

in Ω in the sense of distributions. Furthermore, this transition respects multiplicities.

The discreteness of non-scattering energies will therefore follow from the following theorem.

Theorem 18. *The set of real numbers λ for which the equation (1) has a nontrivial $H_0^2(\Omega; e^{\gamma\langle\cdot\rangle})$-solution is a discrete subset of $[0, \infty[$. For each such λ the space of solutions is finite dimensional.*

The operator on the left-hand side of (1) can be treated nicely via quadratic forms, and for this purpose we define for each $\lambda \in \mathbb{C}$ the quadratic form

$$Q_\lambda = u \longmapsto \left\langle (-\Delta + V - \overline{\lambda})\,u \,\middle|\, \frac{1}{V}\,(-\Delta - \lambda)\,u \right\rangle : H_0^2(\Omega; e^{\gamma\langle\cdot\rangle}) \longrightarrow \mathbb{C},$$

where the L^2-inner product is linear in the second argument. The family $\langle Q_\lambda \rangle_{\lambda\in\mathbb{C}}$ has the pleasant properties enumerated in the theorem below. These properties are analogous to a part of Theorem 4 of [46].

Theorem 19.
1. *The quadratic forms Q_λ form an entire self-adjoint analytic family of forms of type (a) with compact resolvent, and therefore gives rise to a family of operators T_λ, which is an entire self-adjoint analytic family of operators of type (B) with compact resolvent.*
2. *Furthermore, there exists a sequence $\langle \mu_\nu(\cdot) \rangle_{\nu=1}^{\infty}$ of real-analytic functions $\mu_\nu(\cdot) \colon \mathbb{R} \longrightarrow \mathbb{R}$ such that, for real λ, the spectrum of T_λ, which consists of a discrete set of real eigenvalues of finite multiplicity, consists of $\mu_1(\lambda)$, $\mu_2(\lambda)$, …, including multiplicity.*

3. *In addition, for any given $T \in \mathbb{R}_+$, there exists constant $c \in \mathbb{R}_+$ such that*

$$\left|\mu_\nu(\lambda) - \mu_\nu(0)\right| \ll_T e^{c|\lambda|} - 1$$

for all $\lambda \in [-T, T]$ and each $\nu \in \mathbb{Z}_+$.

4. *The pairs $\langle \lambda, u \rangle \in \mathbb{R} \times H_0^2(\Omega; e^{\gamma\langle\cdot\rangle})$ for which (1) holds, are in bijective correspondence with the pairs $\langle \nu, \lambda \rangle \in \mathbb{Z}_+ \times \mathbb{R}$ for which $\mu_\nu(\lambda) = 0$.*

Theorem 18 follows easily from these properties of Q_λ: It is obvious that zero is not an eigenvalue of T_λ for any negative real λ, as $Q_\lambda(u) > 0$ for all non-zero functions $u \in \operatorname{Dom} Q_\lambda$. Hence none of the functions $\mu_\nu(\cdot)$ can vanish identically, so that the set of zeroes of each of them is discrete. Why the union of the zero sets can not have an accumulation point follows immediately from the third statement above, which says that the functions $\mu_\nu(\cdot)$ change their values uniformly locally exponentially. That is, when the value of λ changes by a finite amount, only finitely many $\mu_\nu(\cdot)$ will have enough time to drop to zero, and the discreteness has been obtained.

The proof of Theorem 19 depends heavily on the basic theory of quadratic forms and analytic perturbation theory. For an excellent reference on these topics, we recommend the book by Kato [20], in particular its Chapters VI and VII.

As the arguments in [46], the proof of Theorem 4 there to be precise, work verbatim in our case, except for the required weighted inequality, which is given below, we simply refer the reader to [46]. The following weighted inequality replaces Lemma 2 of [46].

Lemma 20. *Let $\gamma \in \mathbb{R}_+$ with $\gamma \ll_n 1$, and let us be given a compact subset $K \subseteq \mathbb{C}$. Then we have, for all $\lambda \in K$ and all $u \in C_c^\infty(\mathbb{R}^n)$, the weighted inequality*

$$\left\|e^{\gamma\langle\cdot\rangle}\, u\right\| + \left\|e^{\gamma\langle\cdot\rangle}\, \nabla u\right\| + \left\|e^{\gamma\langle\cdot\rangle}\, \nabla \otimes \nabla u\right\| \ll_{n,K} \left\|e^{\gamma\langle\cdot\rangle}\, (-\Delta - \lambda)\, u\right\| + \left\|e^{\gamma\langle\cdot\rangle}\, u\right\|.$$

Proof. The elementary inequalities

$$\langle\cdot\rangle^4 \ll \left|4\pi^2\, |\cdot|^2 + 1\right|^2 \ll \left|4\pi^2\, |\cdot|^2 - \lambda\right|^2 + |\lambda + 1|^2$$

imply that

$$\left\|u\right\| + \left\|\nabla u\right\| + \left\|\nabla \otimes \nabla u\right\| \ll_n \left\|(-\Delta - \lambda)\, u\right\| + \langle\lambda\rangle\, \left\|u\right\|.$$

Now we can introduce the exponential weights into this applying Leibniz's rule:

$$\begin{aligned}
&\big\|e^{\gamma\langle\cdot\rangle}\, u\big\| + \big\|e^{\gamma\langle\cdot\rangle}\, \nabla u\big\| + \big\|e^{\gamma\langle\cdot\rangle}\, \nabla\otimes\nabla u\big\| \\
&\quad \ll_n \big\|e^{\gamma\langle\cdot\rangle}\, u\big\| + \big\|\nabla\big(e^{\gamma\langle\cdot\rangle}\, u\big)\big\| + \big\|\nabla\otimes\nabla\big(e^{\gamma\langle\cdot\rangle}\, u\big)\big\| \\
&\qquad + \big\|\big(\nabla e^{\gamma\langle\cdot\rangle}\big)\, u\big\| + \big\|\big(\nabla e^{\gamma\langle\cdot\rangle}\big)\otimes\nabla u\big\| + \big\|\big(\nabla\otimes\nabla e^{\gamma\langle\cdot\rangle}\big)\, u\big\| \\
&\quad \ll_n \big\|(-\Delta-\lambda)\,\big(e^{\gamma\langle\cdot\rangle}\, u\big)\big\| + \langle\lambda\rangle\, \big\|e^{\gamma\langle\cdot\rangle}\, u\big\| \\
&\qquad + \gamma\big\|e^{\gamma\langle\cdot\rangle}\, u\big\| + \gamma\big\|e^{\gamma\langle\cdot\rangle}\, \nabla u\big\| + \big(\gamma+\gamma^2\big)\, \big\|e^{\gamma\langle\cdot\rangle}\, u\big\| \\
&\quad \ll_n \big\|e^{\gamma\langle\cdot\rangle}\,(-\Delta-\lambda)\, u\big\| + \big\|\big(\nabla e^{\gamma\langle\cdot\rangle}\big)\cdot\nabla u\big\| + \big\|\big(\Delta e^{\gamma\langle\cdot\rangle}\big)\, u\big\| \\
&\qquad + \big(\langle\lambda\rangle+\gamma+\gamma^2\big)\, \big\|e^{\gamma\langle\cdot\rangle}\, u\big\| + \gamma\big\|e^{\gamma\langle\cdot\rangle}\, \nabla u\big\| \\
&\quad \ll_n \big\|e^{\gamma\langle\cdot\rangle}\,(-\Delta-\lambda)\, u\big\| + \big(\langle\lambda\rangle+\gamma+\gamma^2\big)\, \big\|e^{\gamma\langle\cdot\rangle}\, u\big\| + \gamma\big\|e^{\gamma\langle\cdot\rangle}\, \nabla u\big\|.
\end{aligned}$$

Finally, the last term may be absorbed to the original left-hand side provided that $\gamma \ll_n 1$.

6 Proof of Theorem 8

We begin with the following analogue of the Paley–Wiener theorem on exponential decay of the Fourier transform for functions defined in $\mathbb{Z}^n$. It could be compared to, say, Theorem IX.13 of [31].

Theorem 21. *Let $f \in \ell^2(\mathbb{Z}^n)$ and let $\gamma_0 \in \mathbb{R}_+$. Then $e^{\gamma\langle\cdot\rangle} f \in \ell^2(\mathbb{Z}^n)$ for all $\gamma \in \,]0,\gamma_0[$* ***if and only if*** *the function $\check{f} \in L^2(\mathbb{T}^n)$ extends to an analytic function in*

$$\big\{z \in \mathbb{T}^n_{\mathbb{C}} \;\big|\; |\Im z| < \gamma_0/(2\pi)\big\}.$$

Proof. First, let $\gamma_0 \in \mathbb{R}_+$ and $f \in \ell^2(\mathbb{Z}^n)$ be such that $e^{\gamma\langle\cdot\rangle} f \in \ell^2(\mathbb{Z}^n)$ for all $\gamma \in \,]0,\gamma_0[$. Fix some $\gamma \in \,]0,\gamma_0[$. Then certainly $e^{\varepsilon\langle\xi\rangle} e^{2\pi\eta\cdot\xi} f(\xi) \in \ell^2(\mathbb{Z}^n)$ for any $\eta \in \mathbb{R}^n$ with $|\eta| \leqslant \gamma/2\pi$, for small $\varepsilon \in \mathbb{R}_+$, and the series

$$\check{f}(z) = \sum_{\xi\in\mathbb{Z}^n} f(\xi)\, e(\xi\cdot z)$$

clearly converges absolutely and uniformly for all $z \in \mathbb{T}^n_{\mathbb{C}}$ with $|\Im z| \leqslant \gamma/2\pi$, and this limit must be analytic in z since each of the terms is.

Next, assume that $f \in \ell^2(\mathbb{Z}^n)$ and $\gamma_0 \in \mathbb{R}_+$ are such that $\check{f} \in L^2(\mathbb{R}^n)$ extends to an analytic function in $\{z \in \mathbb{T}^n_{\mathbb{C}} | |\Im z| < \gamma_0/2\pi\}$. Fix some $\gamma \in \,]0,\gamma_0[$. Now the restriction of $|\check{f}|$ to the compact set $\{z \in \mathbb{T}^n_{\mathbb{C}} | |\Im z| \leqslant \gamma/2\pi\}$ must be uniformly bounded by some constant $C_\gamma \in \mathbb{R}_+$ only depending on γ.

Then, for arbitrary $y \in \mathbb{R}^n$ with $|y| = \gamma/2\pi$, we may estimate, using Cauchy's integral theorem, that for any given $\xi \in \mathbb{Z}^n$,

$$
\begin{aligned}
f(\xi) &= \int\limits_{\mathbb{T}^n} \check{f}(x)\, e(-x \cdot \xi)\, \mathrm{d}x \\
&= \int\limits_{\mathbb{T}^n} \check{f}(x - iy)\, e\big(-(x - iy) \cdot \xi\big)\big)\, \mathrm{d}x \\
&\ll e^{2\pi y \cdot \xi} \int\limits_{\mathbb{T}^n} \big|\check{f}(x - iy)\big|\, \mathrm{d}x \ll e^{2\pi y \cdot \xi}\, C_\gamma.
\end{aligned}
$$

Thus, we have

$$
f(\xi) \ll_\gamma \inf_{\substack{y \in \mathbb{R}^n, \\ |y| = \gamma/2\pi}} e^{2\pi y \cdot \xi} = e^{-\gamma|\xi|}.
$$

The following is, more or less, a discrete analogue of Theorem 9.

Theorem 22. *Let $f \in \ell^2(\mathbb{Z}^n)$ be such that $e^{\gamma\langle\cdot\rangle} f \in \ell^2(\mathbb{Z}^n)$ for all $\gamma \in \mathbb{R}_+$. Also, let $u\colon \mathbb{Z}^n \longrightarrow \mathbb{C}$ be such that*

$$
\frac{1}{R} \sum_{|\xi| \leqslant R} \big|u(\xi)\big|^2 \longrightarrow 0
$$

as $R \longrightarrow \infty$. Finally, let $\lambda \in \,]0, n[$, and assume that

$$
(-\Delta_{\mathrm{disc}} - \lambda)\, u = f
$$

in $\mathbb{Z}^n$. Then also $e^{\gamma\langle\cdot\rangle} u \in \ell^2(\mathbb{Z}^n)$ for all $\gamma \in \mathbb{R}_+$.

This follows easily: By Theorem 21 the Fourier series $\check{f}$ extends to an entire function in $\mathbb{T}^n_{\mathbb{C}}$. Furthermore,

$$
\big(h(x) - \lambda\big)\, \check{u} = \check{f}
$$

for $x \in \mathbb{T}^n$, where

$$
h(x) = \sum_{j=1}^{n} \sin^2 \frac{x_j}{2}.
$$

We point out that even though u doesn't strictly speaking belong to ℓ^2, it is certainly at most polynomially growing, allowing us to consider $\check{u}$ as a distribution; for more on this point of view, see e.g. Chapter 3 in the book [36].

Now the arguments of Section 4 of [19] show that $\check{u}$ extends to an entire function in $\mathbb{T}^n_{\mathbb{C}}$. Namely, the arguments in Section 4.1 do not involve $\check{f}$ at all, and in Section 4.2, the proof of Lemma 4.3 only depends on the smoothness of $\check{f}$, and after Lemma 4.4, when the analytic continuation of $\check{u}$ is obtained, only the analytic continuation of $\check{f}$ is required. Finally, the conclusion follows from Theorem 21.

Proof of Theorem 8. We shall prove that $u(0) = 0$. Given a point $\xi_0 \in C$, the same argument applied to $u(\cdot + \xi_0)$ and $f(\cdot + \xi_0)$ shows that $u(\xi_0) = 0$.

The idea is to apply the definition of Δ_{disc} and the discrete Helmholtz equation in the form:

$$u(\xi) = (2n - 4\lambda)\; u(\xi+e_n) - u(\xi+2e_n) - \sum_{j=1}^{n-1} \big(u(\xi+e_n+e_j) + u(\xi+e_n-e_j)\big). \quad (2)$$

This holds for all $\xi \in C$. Applying this once to $u(0)$ gives $\leqslant 2n$ terms of the form $u(\xi)$ with $\xi \in C$ and $1 \leqslant \xi_n \leqslant 2$, with constant coefficients, each of which has absolute value $\leqslant 2n$.

Applying (2) again to all the terms of the previous step gives rise to $\leqslant 4n^2$ terms of the form $u(\xi)$ with $\xi \in C$ and $2 \leqslant \xi_n \leqslant 4$, with coefficients of size $\leqslant 4n^2$.

Continuing in this manner, after $N \in \mathbb{Z}_+$ steps $u(0)$ has been represented as the sum of $\leqslant (2n)^N$ terms of the form $u(\xi)$ with $\xi \in C$ and $N \leqslant \xi_n \leqslant 2N$, with coefficients of size $\leqslant (2n)^N$. Thus, by the triangle inequality,

$$|u(0)| \leqslant (2n)^N\, (2n)^N \max_{\substack{\xi \in C,\\ N \leqslant \xi_n \leqslant 2N}} |u(\xi)|\,.$$

Theorem 22 tells us that $u(\xi) \ll_\gamma e^{-\gamma\langle\xi\rangle}$ for all $\xi \in \mathbb{Z}^n$ and any fixed $\gamma \in \mathbb{R}_+$. In particular,

$$u(0) \ll_\gamma (4n^2)^N\, e^{-\gamma N},$$

and choosing $\gamma > \log 4n^2$ and letting $N \longrightarrow \infty$ gives the desired result.

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References

[1] Adams, R.: *Capacity and compact imbeddings*, Journal of Mathematics and Mechanics, 19 (1970), 923–929.

[2] Adams, R., and J. Fournier: *Sobolev Spaces*, Pure and Applied Mathematics Series, Elsevier, 2003.

[3] Agmon, S.: *Spectral properties of Schrödinger operators and scattering theory*, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze, Serie IV, 2 (1975), 151–218.

[4] Agmon, S., and L. Hörmander: *Asymptotic properties of solutions of differential equations with simple characteristics*, Journal d'analyse mathématique, 30 (1976), 1–30.

[5] Blåsten, E., L. Päivärinta, and J. Sylvester: *Corners always scatter*, Communications in Mathematical Physics, April (2014), 1–29.

[6] Cakoni, F., D. Gintides, and H. Haddar: *The existence of an infinite discrete set of transmission eigenvalues*, SIAM Journal on Mathematical Analysis, 42 (2010), 237-255.

[7] Cakoni, F., and H. Haddar: *Transmission eigenvalues in inverse scattering theory*, in [44], 529–578.

[8] Cakoni, F., and H. Haddar: *Transmission eigenvalues*, Inverse Problems, 29 (2013), 100201, 1–3.

[9] Colton, D., A. Kirsch, and L. Päivärinta: *Far field patterns for acoustic waves in an inhomogeneous medium*, SIAM Journal on Mathematical Analysis, 20 (1989), 1472–1482.

[10] Colton, D., and R. Kress: *Inverse Acoustic and Electromagnetic Scattering Theory*, Applied Mathematical Sciences, 93, Springer, 2013.

[11] Colton, D., and P. Monk: *The inverse scattering problem for time-harmonic acoustic waves in an inhomogeneous medium*, The Quarterly Journal of Mechanics and Applied Mathematics, 41 (1988), 97–125.

[12] Colton, D., L. Päivärinta and J. Sylvester: *The interior transmission problem*, Inverse Problems and Imaging, 1 (2007), 13–28.

[13] Grinevich, P. G., and S. V. Manakov: *The inverse scattering problem for the two-dimensional Schrödinger operator, the $\overline{\partial}$-method and non-linear equations*, Functional Analysis and its Applications, 20 (1986), 94–103.

[14] Grinevich, P. G., and R. G. Novikov: *Transparent potentials at fixed energy in dimension two. Fixed-energy dispersion relations for the fast decaying potentials*, Communications in Mathematical Physics, 174 (1995), 409–446.

[15] Hickmann, K.: *Interior transmission eigenvalue problem with refractive index having C^2-transition to the background medium*, Applicable Analysis, 91 (2012), 1675–1690.

[16] Hitrik, M., K. Krupchyk, P. Ola, and L. Päivärinta: *Transmission eigenvalues for operators with constant coefficients*, SIAM Journal on Mathematical Analysis, 42 (2010), 2965–2986.

[17] Hörmander, L.: *Lower bounds at infinity for solutions of differential equations with constant coefficients*, Israel Journal of Mathematics, 16 (1973), 103–116.

[18] Hörmander, L.: *The Analysis of Linear Partial Differential Operators II: Differential Operators with Constant Coefficients*, Classics in Mathematics, Springer, 2005.

[19] Isozaki, H., and H. Morioka: *A Rellich type theorem for discrete Schrödinger operators*, arXiv:1208.4428 [math.SP].

[20] Kato, T.: *Perturbation Theory for Linear Operators*, Classics in Mathematics, Springer, 1995.

[21] Kirsch, A.: *The denseness of the far field patterns for the transmission problem*, IMA Journal of Applied Mathematics, 37 (1986), 213–225.

[22] Lakshtanov, E., and B. Vainberg: *Applications of elliptic operator theory to the isotropic interior transmission eigenvalue problem*, Inverse Problems, 29 (2013), 104003, 1–19.

[23] Littman, W.: *Decay at infinity of solutions to partial differential equations with constant coefficients*, Transactions of the American Mathematical Society, 123 (1966), 449-459.

[24] Littman, W.: *Decay at infinity of solutions to partial differential equations; removal of the curvature assumption*, Israel Journal of Mathematics, 8 (1970), 403–407.

[25] Littman, W.: *Maximal rates of decay of solutions of partial differential equations*, Archive for Rational Mechanics and Analysis, 37 (1970), 11–20.

[26] Murata, M.: *A theorem of Liouville type for partial differential equations with constant coefficients*, Journal of the Faculty of Science, the University of Tokyo, Section IA Mathematics, 21 (1974), 395–404.

[27] Murata, M.: *Asymptotic behaviors at infinity of solutions to certain partial differential equations*, Journal of the Faculty of Science, the University of Tokyo, Section IA Mathematics, 23 (1976), 107–148.

[28] Newton, R. G.: *Construction of potentials from the phase shifts at fixed energy*, Journal of Mathematical Physics, 3 (1962), 75–82.

[29] Päivärinta, L., M. Salo, and G. Uhlmann: *Inverse scattering for the magnetic Schrödinger operator*, Journal of Functional Analysis, 259 (2010), 1771–1798.

[30] Päivärinta, L., and J. Sylvester: *Transmission eigenvalues*, SIAM Journal on Mathematical Analysis, 40 (2008), 738–753.

[31] Reed, M., and B. Simon: *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness*, Academid Press, 1975.

[32] Regge, T.: *Introduction to complex orbital moments*, Il Nuovo Cimento, 14 (1959), 951–976.

[33] Rellich, F.: *Über das asymptotische Verhalten der Lösungen von $\Delta u + \lambda u = 0$ im unendlichen Gebieten*, Jahresbericht der Deutschen Mathematiker-Vereinigung, 53 (1943), 57–65.

[34] Robbiano, L.: *Spectral analysis of the interior transmission eigenvalue problem*, Inverse Problems, 29 (2013), 104001, 1–28.

[35] Rudin, W.: *Real and Complex Analysis*, International Series in Pure and Applied Mathematics, McGraw-Hill, 1986.

[36] Ruzhansky, M., and V. Turunen: *Pseudo-Differential Operators and Symmetries. Background Analysis and Advanced Topics*, Pseudo-Differential Operators, Theory and Applications, 2, Birkhäuser, 2010.

[37] Sabatier, P. C.: *Asymptotic properties of the potentials in the inverse-scattering problem at fixed energy*, Journal of Mathematical Physics, 7 (1966), 1515–1531.

[38] Serov, V.: *Transmission eigenvalues for non-regular cases*, Communications in Mathematical Analysis, 14 (2013), 129–142.

[39] Serov, V., and J. Sylvester: *Transmission eigenvalues for degenerate and singular cases*, Inverse Problems, 28 (2012), 065004, 1–8.

[40] Shaban, W., and B. Vainberg: *Radiation conditions for the difference Schrödinger operators*, Applicable Analysis, 80 (2001), 525–556.

[41] Sylvester, J.: *Discreteness of transmission eigenvalues via upper triangular compact operators*, SIAM Journal on Mathematical Analysis, 44 (2012), 341–354.

[42] Sylvester, J., and G. Uhlmann: *A global uniqueness theorem for an inverse boundary value problem*, Annals of Mathematics, 125 (1987), 153–169.

[43] Trèves, F.: *Differential polynomials and decay at infinity*, Bulletin of the American Mathematical Society (New Series), 66 (1960), 184–186.

[44] Uhlmann, G. (editor): *Inverse Problems and Applications: Inside Out II*, MSRI Publications 60, Cambridge University Press, 2013.

[45] Vekua, I. N.: *Metaharmonic functions*, Trudy Tbilisskogo matematicheskogo instituta, 12 (1943), 105–174.

[46] Vesalainen, E. V.: *Transmission eigenvalues for a class of non-compactly supported potentials*, Inverse Problems, 29 (2013), 104006, 1–11.

[47] Wong, M. W.: *An Introduction to Pseudo-Differential Operators*, World Scientific, 1999.

III. Strictly Convex Corners Scatter

Lassi Päivärinta[*], *Mikko Salo*[†], *Esa V. Vesalainen*[‡]

Abstract

We prove the absence of non-scattering energies for potentials in the plane having a corner of angle smaller than π. This extends the earlier result of Blåsten, Päivärinta and Sylvester who considered rectangular corners. In three dimensions, we prove a similar result for any potential with a circular conic corner whose opening angle is outside a countable subset of $(0, \pi)$.

1 Introduction

1.1 Background

This article is concerned with *non-scattering energies*. These are energies $\lambda > 0$ for which there exists a nontrivial incident wave that does not scatter (equivalently, the far field operator at energy $\lambda > 0$ has nontrivial kernel).

Certain reconstruction methods in inverse scattering theory, such as the linear sampling method [CK96] or the factorization method [KN08], may fail at non-scattering energies and therefore these energies are to be avoided. This has led people to study the usually larger class of interior transmission eigenvalues which first appeared in [CM88, Ki86]. For acoustic scattering, the transmission eigenvalues often form an infinite discrete set [PS08, CGH10], and in recent years they have been studied intensively. For more information about transmission eigenvalues, we recommend the survey [CH13a] as well as the articles mentioned in the recent editorial [CH13b] and their references.

Results on non-scattering energies appear to be scarce, apart from discreteness results which follow from corresponding results for transmission eigenvalues. For radial compactly supported potentials, the set of non-scattering energies coincides with the infinite discrete set of transmission eigenvalues [CM88]. Non-scattering energies are exactly those energies for which the scattering matrix has

[*]Department of Mathematics and Statistics, University of Helsinki
[†]Department of Mathematics and Statistics, University of Jyväskylä
[‡]Department of Mathematics and Statistics, University of Helsinki

1 as an eigenvalue, and [GH12] constructs C_c^∞ potentials with no non-scattering energies.

It is relevant to mention here the related topic of transparent potentials. These are nonzero potentials whose far field operator is identically zero at some fixed energy λ, and thus no incident wave with energy λ scatters. Several constructions of such radial potentials have been given, see e.g. the works [Re59, Ne62, Sa66, GM86, GN95].

The recent work [BPS14] suggests that corner points in the scattering potential always generate a scattered wave. More precisely, [BPS14] proves the absence of non-scattering energies in acoustic scattering for certain contrasts having corners with a right angle, i.e. for contrasts supported in a rectangle K that do not vanish at a corner point of K. In this paper we extend this result to corners with arbitrary opening angle $< \pi$ in two dimensions. In three dimensions, we prove that circular conic corner points with angle outside an at most countable subset of $(0, \pi)$ lead to absence of non-scattering energies. By the "opening angle" of a cone

$$\big\{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \;\big|\; |x'| \leq c x_n\big\},$$

say, where $c \in \mathbb{R}_+$, we mean the angle $\vartheta \in (0, \pi)$ such that $\tan(\vartheta/2) = c$.

1.2 Non-scattering energies

Let us state the precise definition of non-scattering energies. This notion makes sense in the context of short range scattering theory in $\mathbb{R}^n$, and we will formulate our results using the notation of quantum mechanical scattering following [Hö83, Chapter XIV]. We will discuss later the analogous case of acoustic scattering, where the term *non-scattering wavenumbers* is more appropriate.

Let V be a short range potential, which in this paper will mean that V is in $L^\infty(\mathbb{R}^n)$ and there are $C > 0$, $\varepsilon > 0$ such that

$$|V(x)| \leq C\langle x\rangle^{-1-\varepsilon} \quad \text{a.e. in } \mathbb{R}^n.$$

Here we write $\langle x\rangle = (1+|x|^2)^{1/2}$. For any $\lambda > 0$ and $g \in L^2(S^{n-1})$, we consider the *incident wave*

$$u_0(x) = \int_{S^{n-1}} e^{i\sqrt{\lambda}x\cdot\omega} g(\omega)\, d\omega, \tag{1.1}$$

which solves the free Schrödinger equation $(-\Delta-\lambda)u_0 = 0$ in $\mathbb{R}^n$. Corresponding to the incident wave u_0, there is a unique solution of the perturbed Schrödinger equation

$$(-\Delta + V - \lambda)u = 0 \quad \text{in } \mathbb{R}^n$$

having the form

$$u = u_0 + v,$$

where v satisfies the *outgoing radiation condition.* There are many equivalent formulations of this condition that selects the unique outgoing solution of the Schrödinger equation: we follow [Hö83] and say that v satisfies the outgoing radiation condition if $v = (-\Delta - \lambda - i0)^{-1} f$ for some f in the Agmon–Hörmander space B (see Section 2 for the precise definitions). The function v is called the *outgoing scattered wave* corresponding to u_0, and u is called the *total wave.*

If u_0 corresponds to $g \in L^2(S^{n-1})$ as above and if $x = r\theta$ where $\theta \in S^{n-1}$, then u_0 has the following asymptotics as $r \to \infty$:

$$u_0(r\theta) \sim c r^{-\frac{n-1}{2}} (e^{i\sqrt{\lambda} r} g(\theta) + i^{n-1} e^{-i\sqrt{\lambda} r} g(-\theta)).$$

The scattered wave v has the asymptotics

$$v(r\theta) \sim c r^{-\frac{n-1}{2}} e^{i\sqrt{\lambda} r} A_\lambda g(\theta)$$

where A_λ is the *far field operator*, which is a bounded linear operator

$$A_\lambda : L^2(S^{n-1}) \to L^2(S^{n-1}).$$

The function $A_\lambda g$ is called the *far field pattern* of the scattered wave v. If the far field pattern vanishes and additionally V is compactly supported, the Rellich uniqueness theorem (see [Ve14] for references) implies that also the scattered wave v must be compactly supported. Thus the vanishing of the far field pattern may be interpreted so that the incident wave u_0 does not produce any scattered wave at infinity.

We may then divide all energies $\lambda > 0$ in two classes: those for which all nontrivial incident waves scatter, and those for which there exist nontrivial incident waves that cannot be observed at infinity. The latter case is the case of non-scattering energies:

Definition. Let V be a short range potential in $\mathbb{R}^n$. We say that $\lambda > 0$ is a non-scattering energy for the potential V, if there exists a nonzero $g \in L^2(S^{n-1})$ for which $A_\lambda g = 0$.

1.3 Main results

Our argument for the absence of non-scattering energies is based on suitable complex geometrical optics solutions to the Schrödinger equation. Since these solutions grow exponentially at infinity, it will be natural to assume that the potentials satisfy a corresponding decay condition.

Definition. $V \in L^\infty(\mathbb{R}^n)$ is called *superexponentially decaying* if for any $\gamma > 0$ there is $C_\gamma > 0$ such that $|V(x)| \leq C_\gamma e^{-\gamma|x|}$ a.e. in $\mathbb{R}^n$.

The main results of this paper are as follows. Below we will write χ_E for the characteristic function of a set E and $C^s(\mathbb{R}^n)$ for the Hölder spaces with norm (for $0 < s < 1$)

$$\|f\|_{C^s} = \|f\|_{L^\infty} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^s}.$$

Theorem 1.1. *Let $V(x) = \chi_C(x)\varphi(x)$ where $C \subset \mathbb{R}^2$ is a closed strictly convex cone with its vertex at the origin, and φ is a superexponentially decaying function in $\mathbb{R}^2$ such that $\langle x \rangle^\alpha \varphi \in C^s(\mathbb{R}^2)$ for some $\alpha > 5/3$ and $s > 0$. Let also $\varphi(0) \neq 0$. Then there are no non-scattering energies for the potential V.*

Theorem 1.2. *Let $V(x) = \chi_C(x)\varphi(x)$ where $C \subset \mathbb{R}^3$ is a closed circular cone with opening angle $\gamma \in (0, \pi)$ having its vertex at the origin, and φ is a superexponentially decaying function in $\mathbb{R}^3$ such that $\langle x \rangle^\alpha \varphi \in C^s(\mathbb{R}^3)$ for some $\alpha > 9/4$ and $s > 1/4$. Let also $\varphi(0) \neq 0$.*

There exists an at most countable subset $E \subset (0, \pi)$ such that if V is as above and if $\gamma \in (0, \pi) \setminus E$, then there are no non-scattering energies for the potential V.

Remark. The technical conditions for φ in the above theorems are satisfied for instance if φ is a compactly supported s-Hölder continuous function in $\mathbb{R}^n$ where $s > 0$ if $n = 2$ or $s > 1/4$ if $n = 3$.

The above theorems also imply analogous statements in acoustic scattering. In this case we consider $\mathbb{R}^n$ as a medium where acoustic waves propagate, and the refractive index of the medium is assumed to be $(1+m)^{1/2}$ where the contrast m satisfies the short range condition (one often writes $n^2 = 1+m$ where n is the refractive index). Let $k > 0$ be a wavenumber, and write $\lambda = k^2$. We consider an incident wave u_0 as in (1.1) solving $(-\Delta - k^2)u_0 = 0$ in $\mathbb{R}^n$. There is a corresponding total wave $u = u_0 + v$ that solves

$$(-\Delta - k^2(1+m))u = 0 \quad \text{in } \mathbb{R}^n,$$

where the scattered wave v satisfies the outgoing radiation condition.

Now, we say that $k > 0$ is a non-scattering wavenumber for the contrast m if there is a nontrivial incident wave u_0 such that the scattered wave v has trivial asymptotics at infinity. The proofs of Theorems 1.1 and 1.2 apply in this situation, and we obtain corresponding results which state the absence of non-scattering wavenumbers for contrasts m that satisfy exactly the same conditions as the potentials V.

We mention that the method of complex geometrical optics solutions that is used for proving the above theorems has a long history both in inverse scattering problems [NK87, Na88, No88, Ra88] and inverse boundary value problems [Ca80, SU87]. See [No08, Uh09] for further references.

1.4 Structure of argument

We follow the approach of [BPS14] and argue by contradiction. Assume that $\lambda \in \mathbb{R}_+$ is a non-scattering energy for the potential V. Then, we have nontrivial solutions $w, v_0 \in B_2^*$ to

$$(-\Delta + V - \lambda)\, w = 0, \quad \text{and} \quad (-\Delta - \lambda)\, v_0 = 0$$

in $\mathbb{R}^n$, satisfying $w - v_0 \in \mathring{B}_2^*$. Since v_0 is real analytic, the lowest degree non-vanishing terms in its Taylor series at the origin form a harmonic homogeneous polynomial $H(x) \not\equiv 0$ of degree $N \geq 0$. The desired contradiction will be obtained by showing that $H(x) \equiv 0$.

It is proved in Section 2 that for non-scattering energies λ,

$$\int_{\mathbb{R}^n} V u v_0 = 0$$

for any $u \in e^{\gamma\langle\cdot\rangle} L^2(\mathbb{R}^n)$ solving $(-\Delta + V - \lambda)\, u = 0$, where $\gamma \in \mathbb{R}_+$ is arbitrary.

In Section 3, we shall discuss the existence of solutions of the form $u = e^{-\rho\cdot x}\,(1 + \psi)$ to $(-\Delta + V - \lambda)\, u = 0$ for $\rho \in \mathbb{C}^n$ with $\rho \cdot \rho = -\lambda$ and ψ being well controlled as $|\rho| \longrightarrow \infty$. In Section 4 we show that substituting the complex geometrical optics to the above integral identity implies the vanishing of a certain Laplace transform. More precisely, after some estimations we see that

$$\int_C e^{-\rho\cdot x}\, H(x)\, dx \lesssim |\rho|^{-N-2-\beta},$$

for some small $\beta \in \mathbb{R}_+$, as $|\rho| \longrightarrow \infty$, and we restrict to a suitable subset of vectors $\rho \in \mathbb{C}^n$ with $\rho \cdot \rho = -\lambda$. On the other hand, from the homogeneity of $H(x)$, we see that

$$\int_C e^{-\rho\cdot x}\, H(x)\, dx = |\rho|^{-N-2} \int_C e^{-\rho/|\rho|\cdot x}\, H(x)\, dx,$$

for the same ρ as before. The last two estimates turn out to be compatible only if

$$\int_C e^{-\rho\cdot x}\, H(x)\, dx = 0$$

for certain vectors $\rho \in \mathbb{C}^n$ with $\rho \cdot \rho = 0$ (as opposed to $\rho \cdot \rho = -\lambda$). The last identity asserts the vanishing of the Laplace transform of $\chi_C H$, where H is a harmonic homogeneous polynomial and χ_C is the characteristic function of the cone C, for certain complex vectors.

Up to this point, we have closely followed the approach of [BPS14]. We now depart from this approach and move to polar coordinates in the Laplace transform identity. This implies the vanishing of the following integrals over a spherical cap for certain $\rho \in \mathbb{C}^n$:

$$\int_{C \cap S^{n-1}} (\rho \cdot x)^{-N-n} H(x) \, dS(x) = 0.$$

As a restriction of a harmonic homogeneous polynomial, H can be expanded in terms of spherical harmonics of fixed degree. The main difference between our approach and [BPS14] is that we will perform computations in terms of these spherical harmonics.

In Section 5 we discuss the case $n = 2$, which is particularly simple. There $H(x)$ must be of the form $a(x + iy)^N + b(x - iy)^N$ for some constants $a, b \in \mathbb{C}$. When this is inserted in the above vanishing relation, the ensuing integrals can be evaluated explicitly and one obtains a concrete homogeneous linear pair of equations for a and b. It is not difficult to prove that the corresponding determinant is nonzero, and so we conclude that $a = b = 0$ and $H(x) \equiv 0$ as desired.

Section 6 considers the three-dimensional case. The polynomial $H(x)$ can be written as a finite linear combination of spherical harmonics, and one can again obtain a homogeneous linear system; this time the "unknowns" are the constant coefficients multiplied by certain concrete but complicated integrals, and the determinant of the system can be arranged to be a Vandermonde determinant. Thus, the vanishing of the coefficients of $H(x)$ is reduced to proving that all of these complicated integrals are nonzero. It is not clear to us how to do so. However, the integrals depend analytically on the opening angle, and we can prove that they are not identically zero as functions of the opening angle. In this way we get the desired contradiction when the opening angle is outside some at most countable set of exceptional angles.

We remark that there are two complications in extending the methods to dimensions $n \geq 4$. First of all, the construction of complex geometrical optics solutions is carried out by a Neumann series argument where the potential V appears as a pointwise multiplier. The fact that V is not very regular (it is essentially the characteristic function of a cone) implies that our construction of complex geometrical optics solutions, which is based on L^p estimates from [KRS87] that were also used in an early version of [BPS14], only gives good enough estimates when $n = 2, 3$. In [BPS14] another construction of solutions was employed, this construction works when $n \geq 2$ but seems to apply to "polygonal" cones instead of the circular cones that we use. The second complication is related to the more complex structure of spherical harmonics in high dimensions, which makes the resulting integrals difficult to evaluate.

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2 Short range scattering

In this section we recall some basic facts in short range scattering theory that are required for the setup in this paper. The following proposition is the main result in this section, and only its statement will be used in the subsequent sections.

Proposition 2.1. *Let V be a superexponentially decaying potential. If $\lambda > 0$ is a non-scattering energy for V, then*

$$\int_{\mathbb{R}^n} V u v_0 \, dx = 0$$

for some nontrivial solution $v_0 \in B^*$ of $(-\Delta - \lambda)v_0 = 0$ in $\mathbb{R}^n$, and for all $u \in L^2_{loc}(\mathbb{R}^n)$ such that $(-\Delta + V - \lambda)u = 0$ in $\mathbb{R}^n$ and $u \in e^{\gamma\langle x\rangle} L^2(\mathbb{R}^n)$ for some $\gamma > 0$.

The results in this section are stated in terms of Agmon–Hörmander spaces B and B^*. The basic reference is [Hö83, Chapter XIV]. However, most of the next results are also contained in [PSU10] (see also [Me95], [UV02]) in a convenient form. Thus the reader may refer to [PSU10] for proofs and further details on the statements in this section.

2.1 Function spaces

The space B (see [Hö83, Section 14.1]) is the set of those $u \in L^2(\mathbb{R}^n)$ for which the norm

$$\|u\|_B = \sum_{j=1}^{\infty} (2^{j-1} \int_{X_j} |u|^2 \, dx)^{1/2}$$

is finite. Here $X_1 = \{|x| < 1\}$ and $X_j = \{2^{j-2} < |x| < 2^{j-1}\}$ for $j \geq 2$. This is a Banach space whose dual B^* consists of all $u \in L^2_{\text{loc}}(\mathbb{R}^n)$ such that

$$\|u\|_{B^*} = \sup_{R>1} \left[\frac{1}{R} \int_{|x|<R} |u|^2 \, dx \right]^{1/2} < \infty.$$

The set $C_c^\infty(\mathbb{R}^n)$ is dense in B but not in B^*. The closure in B^* is denoted by $\mathring{B}^*$, and $u \in B^*$ belongs to $\mathring{B}^*$ if and only if

$$\lim_{R\to\infty} \frac{1}{R} \int_{|x|<R} |u|^2 \, dx = 0.$$

We will also need the Sobolev space variant B_2^* of B^*, defined via the norm

$$\|u\|_{B_2^*} = \sum_{|\alpha|\leq 2} \|D^\alpha u\|_{B^*}.$$

If $\lambda > 0$ we will consider the sphere $M_\lambda = \{\xi \in \mathbb{R}^n \, ; \, |\xi| = \sqrt{\lambda}\}$ with Euclidean surface measure dS_λ. The corresponding L^2 space is $L^2(M_\lambda) = L^2(M_\lambda, dS_\lambda)$, and of course $L^2(S^{n-1}) = L^2(M_1)$.

2.2 Scattering solutions

We consider scattering in $\mathbb{R}^n$ with respect to incident waves u_0 with fixed energy $\lambda > 0$, where u_0 has the form

$$u_0 = P_0(\lambda) g, \qquad g \in L^2(M_\lambda).$$

Here $P_0(\lambda)$ is the *Poisson operator*

$$P_0(\lambda) g(x) = \frac{i}{(2\pi)^{n-1}} \int_{M_\lambda} e^{i x\cdot\xi} g(\xi) \, \frac{dS_\lambda(\xi)}{2\sqrt{\lambda}}, \qquad x \in \mathbb{R}^n.$$

Thus u_0 is a Herglotz wave corresponding to a pattern g at infinity. The function u_0 belongs to B_2^* and it satisfies the Helmholtz equation

$$(-\Delta - \lambda) u_0 = 0 \qquad \text{in } \mathbb{R}^n.$$

Now consider quantum scattering in $\mathbb{R}^n$ where the medium properties are described by a short range potential V. By this we mean that $V \in L^\infty(\mathbb{R}^n)$ is real valued, and for some $C > 0, \varepsilon > 0$ one has

$$|V(x)| \leq C\langle x\rangle^{-1-\varepsilon} \quad \text{for a.e. } x \in \mathbb{R}^n.$$

The outgoing resolvent $R_V(\lambda + i0) = (-\Delta + V - \lambda - i0)^{-1}$ is well defined for all $\lambda > 0$, and it is a bounded operator

$$R_V(\lambda + i0) : B \to B_2^*.$$

For any incoming wave $u_0 = P_0(\lambda)g$ where $g \in L^2(M_\lambda)$, there is a unique total wave u solving the equation

$$(-\Delta + V - \lambda)u = 0 \qquad \text{in } \mathbb{R}^n$$

such that $u = u_0 + v$ where v is *outgoing* (meaning that $v = R_0(\lambda + i0)f$ for some $f \in B$). In fact, if $u_0 = P_0(\lambda)g$, then one has

$$u = P_V(\lambda)g$$

where $P_V(\lambda) : L^2(M_\lambda) \to B_2^*(\mathbb{R}^n)$ is the outgoing Poisson operator

$$P_V(\lambda)g = P_0(\lambda)g - R_V(\lambda + i0)(VP_0(\lambda)g).$$

2.3 Asymptotics

We write $u \sim u_0$ to denote that $u - u_0 \in \mathring{B}^*$, which is interpreted so that u and u_0 have the same asymptotics at infinity. Now if $g \in L^2(M_\lambda)$, then $P_V(\lambda)g$ has asymptotics

$$P_V(\lambda)g \sim c_\lambda r^{-\frac{n-1}{2}} \left[e^{i\sqrt{\lambda}r}(S_V(\lambda)g)(\sqrt{\lambda}\theta) + i^{n-1}e^{-i\sqrt{\lambda}r}g(-\sqrt{\lambda}\theta)\right]$$

as $r = |x| \to \infty$, where $x = r\theta$ and $c_\lambda = (\sqrt{\lambda}/2\pi i)^{\frac{n-3}{2}}/4\pi$. Here

$$S_V(\lambda) : L^2(M_\lambda) \to L^2(M_\lambda)$$

is the *scattering matrix* for V at energy λ. It is a unitary operator, $S_V(\lambda)^*S_V(\lambda) = \mathrm{Id}$, and if $V = 0$ one has $S_0(\lambda) = \mathrm{Id}$. The operator

$$A_V(\lambda) = S_V(\lambda) - S_0(\lambda) : L^2(M_\lambda) \to L^2(M_\lambda)$$

is called the *far field operator*, and $A_V(\lambda)g$ is the *far field pattern* of the outgoing scattered wave v at infinity.

Recall from the introduction that if $\lambda > 0$ is such that there exists a nontrivial Herglotz wave $v_0 = P_0(\lambda)g$ for which the far field pattern $A_V(\lambda)g$ is identically zero, we say that λ is a non-scattering energy. Thus, λ is a non-scattering energy if and only if there is a nontrivial function $g \in L^2(M_\lambda)$ such that

$$(S_V(\lambda) - S_0(\lambda))g = 0.$$

2.4 Orthogonality identities

We now recall the "boundary pairing" for scattering solutions [Me95], [PSU10, Proposition 2.3].

Proposition 2.2. *Let $u, v \in B^*$ and $(H_0 - \lambda)u \in B, (H_0 - \lambda)v \in B$. If u and v have the asymptotics*

$$u \sim r^{-\frac{n-1}{2}} \left[e^{i\sqrt{\lambda}r} g_+(\sqrt{\lambda}\theta) + e^{-i\sqrt{\lambda}r} g_-(-\sqrt{\lambda}\theta) \right],$$
$$v \sim r^{-\frac{n-1}{2}} \left[e^{i\sqrt{\lambda}r} h_+(\sqrt{\lambda}\theta) + e^{-i\sqrt{\lambda}r} h_-(-\sqrt{\lambda}\theta) \right]$$

for some $g_\pm, h_\pm \in L^2(M_\lambda)$, then

$$(u|(H_0 - \lambda)v)_{\mathbb{R}^n} - ((H_0 - \lambda)u|v)_{\mathbb{R}^n}$$
$$= 2i\lambda^{-\frac{n-2}{2}} \left[(g_+|h_+)_{M_\lambda} - (g_-|h_-)_{M_\lambda} \right].$$

Here $(u|v)_{\mathbb{R}^n} = \int_{\mathbb{R}^n} u\bar{v}\, dx$ and $(g|h)_{M_\lambda} = \int_{M_\lambda} g\bar{h}\, dS_\lambda$.

As a consequence, the existence of a nontrivial $g \in L^2(M_\lambda)$ for which $A_V(\lambda)g = 0$ is characterized by the following orthogonality identity:

Proposition 2.3. *Let V be a short range potential, let $\lambda > 0$, and let $g \in L^2(M_\lambda)$. Then $A_V(\lambda)g = 0$ if and only if for all $f \in L^2(M_\lambda)$ one has*

$$\int_{\mathbb{R}^n} V u v_0 \, dx = 0$$

where $u = P_V(\lambda)f$ and $v_0 = P_0(\lambda)g$.

Proof. Apply Proposition 2.2 with $u = P_V(\lambda)S_V(\lambda)^* f$ and $v = v_0$ to obtain

$$\int_{\mathbb{R}^n} V u v_0 \, dx = 2i\lambda^{-\frac{n-2}{2}} c_\lambda^2 \Big[(S_V(\lambda)S_V(\lambda)^* f|g)_{M_\lambda} - (S_V(\lambda)^* f|g)_{M_\lambda} \Big]$$
$$= -2i\lambda^{-\frac{n-2}{2}} c_\lambda^2 (f|(S_V(\lambda) - S_0(\lambda))g)_{M_\lambda}$$

since $S_V(\lambda)$ is unitary and $S_0(\lambda) = \mathrm{Id}$. Now $A_V(\lambda)g = 0$ if and only if the integral over $\mathbb{R}^n$ vanishes for all $f \in L^2(M_\lambda)$. $\square$

The last proposition shows that if $A_V(\lambda)g = 0$, then the function Vv_0 where $v_0 = P_0(\lambda)g$ is orthogonal to all scattering solutions $P_V(\lambda)f$ where $f \in L^2(M_\lambda)$. It is well known that if V has certain decay properties, then solutions of $(-\Delta + V - \lambda)u = 0$ satisfying corresponding growth conditions can be approximated by scattering solutions. The next result from [UV02] (see also [PSU10, Proposition 2.4]) concerns exponentially decaying potentials.

Proposition 2.4. *Assume that $V \in L^\infty(\mathbb{R}^n)$ satisfies $|V(x)| \leq Ce^{-\gamma_0\langle x\rangle}$ for some $\gamma_0 > 0$. Let $0 < \gamma < \gamma_0$. Given any $u \in e^{\gamma\langle x\rangle}L^2$ with $(-\Delta + V - \lambda)u = 0$, there exist $g_j \in L^2(M_\lambda)$ such that $P_V(\lambda)g_j \to u$ in $e^{\gamma_0\langle x\rangle}L^2$.*

We will later want to use complex geometrical optics solutions u. Since these may have arbitrarily large exponential growth, it is natural to assume that the potential is superexponentially decaying.

Proposition 2.5. *Let V be a superexponentially decaying potential, let $\lambda > 0$, and let $g \in L^2(M_\lambda)$. Then $A_V(\lambda)g = 0$ if and only if one has*

$$\int_{\mathbb{R}^n} Vuv_0 \, dx = 0$$

for $v_0 = P_0(\lambda)g$ and for all $u \in L^2_{loc}(\mathbb{R}^n)$ such that $(-\Delta + V - \lambda)u = 0$ in $\mathbb{R}^n$ and $u \in e^{\gamma\langle x\rangle}L^2(\mathbb{R}^n)$ for some $\gamma > 0$.

Proof. Follows by combining Propositions 2.3 and 2.4. $\square$

Proof of Proposition 2.1. Follows immediately from Proposition 2.5. $\square$

3 Complex geometrical optics solutions

By Proposition 2.1 we know that if V is superexponentially decaying and if $\lambda > 0$ is a non-scattering energy for V, then there exists a nontrivial $v_0 \in B^*$ satisfying $(-\Delta - \lambda)v_0 = 0$ such that we have

$$\int_{\mathbb{R}^n} Vuv_0 \, dx = 0$$

for any exponentially growing solution u of $(-\Delta + V - \lambda)u = 0$.

We will employ complex geometrical optics solutions as the solutions u above. It will be important to have L^q estimates for large q with suitable decay for the remainder term ψ in these solutions. In [BPS14], such solutions were constructed in all dimensions $n \geq 2$ but for "polygonal" cones C (that is, the cross-section of the cone is a polygon). For our higher dimensional result it is more convenient to use circular cones, and it turns out that the argument of [BPS14] is not valid in this case. Therefore we base our construction on certain L^p estimates from [KRS87]. This argument gives sufficient estimates for $n = 2, 3$ but not for $n \geq 4$.

Theorem 3.1. *Let $\lambda > 0$, assume that $n \in \{2, 3\}$, and let $q = \frac{2(n+1)}{n-1}$. Let $V(x) = \chi_C(x)\varphi(x)$ where $C \subset \mathbb{R}^n$ is a closed circular cone with opening angle $< \pi$, and where φ satisfies $\langle x\rangle^\alpha \varphi \in C^s(\mathbb{R}^n)$ for some $\alpha > 5/3$ and $s > 0$ if $n = 2$ (resp. $\alpha > 9/4$ and $s > 1/4$ if $n = 3$).*

If $\rho \in \mathbb{C}^n$ satisfies $\rho \cdot \rho = -\lambda$ and $|\mathrm{Im}(\rho)|$ is sufficiently large, there is a solution of

$$(-\Delta + V - \lambda)u = 0 \quad \textit{in } \mathbb{R}^n$$

of the form $u = e^{-\rho\cdot x}(1 + \psi)$, where ψ satisfies for some $\delta > 0$

$$\|\psi\|_{L^q(\mathbb{R}^n)} = O(|\mathrm{Im}(\rho)|^{-n/q-\delta}) \textit{ as } |\rho| \to \infty.$$

We begin by stating some a priori inequalities. Below we will write $D = -i\nabla$, $\mathscr{S}$ will be the space of Schwartz test functions, and $H^{s,p}$ for $s \in \mathbb{R}$ and $1 < p < \infty$ will be the Bessel potential space with norm $\|u\|_{H^{s,p}} = \|\langle D\rangle^s f\|_{L^p}$ with $\langle D\rangle = (1 + D^2)^{1/2}$. Also, r' will denote the Hölder conjugate exponent of r (so that $1/r + 1/r' = 1$).

Proposition 3.2. *Let $n \geq 2$, let $1 < r < 2$, and assume that*

$$\frac{1}{r} - \frac{1}{r'} \in \begin{cases} \left[\frac{2}{n+1}, \frac{2}{n}\right] & \textit{if } n \geq 3, \\ \left[\frac{2}{n+1}, \frac{2}{n}\right) & \textit{if } n = 2. \end{cases}$$

There is a constant $M > 0$ such that for any $\zeta \in \mathbb{C}^n$ with $\mathrm{Re}(\zeta) \neq 0$, one has

$$\|f\|_{L^{r'}} \leq M|\mathrm{Re}(\zeta)|^{n(1/r-1/r')-2}\|(-\Delta + 2\zeta \cdot D)f\|_{L^r}, \quad f \in \mathscr{S}.$$

Proof. This is a consequence of the uniform Sobolev inequalities in [KRS87]. In particular, the case $n \geq 3$ follows from [KRS87, Theorem 2.4] after dilations and conjugations by the exponentials $e^{\pm i\,\mathrm{Re}(\zeta)\cdot x}$. For the case $n = 2$ and more details, see [Ru02, Section 5.3]. □

The next result shows solvability for an inhomogeneous equation related to complex geometrical optics solutions.

Proposition 3.3. *Let $n \geq 2$, let $1 < r < 2$, and assume that*

$$\frac{1}{r} - \frac{1}{r'} \in \left[\frac{2}{n+1}, \frac{2}{n} \right).$$

Let $s \in \mathbb{R}$, and let V be a measurable function in $\mathbb{R}^n$ satisfying the following multiplier property for some constant $A > 0$:

$$\|Vf\|_{H^{s,r}} \leq A\|f\|_{H^{s,r'}}, \qquad f \in \mathscr{S}.$$

There exist constants $C > 0$ and $R > 0$ such that whenever $\zeta \in \mathbb{C}^n$ satisfies $|\mathrm{Re}(\zeta)| \geq R$, then for any $f \in H^{s,r}(\mathbb{R}^n)$ the equation

$$(-\Delta + 2\zeta \cdot D + V)u = f \quad \text{in } \mathbb{R}^n$$

has a solution $u \in H^{s,r'}(\mathbb{R}^n)$ satisfying

$$\|u\|_{H^{s,r'}} \leq C|\mathrm{Re}(\zeta)|^{n(1/r-1/r')-2}\|f\|_{H^{s,r}}.$$

Proof. Let $\zeta \in \mathbb{C}^n$ with $\mathrm{Re}(\zeta) \neq 0$. We will use a standard duality argument to obtain a solvability result from the a priori estimates in Proposition 3.2. Write $P = (-\Delta + 2\zeta \cdot D)$, so the formal adjoint is $P^* = (-\Delta + 2\bar{\zeta} \cdot D)$. Applying Proposition 3.2 to $\langle D \rangle^{-s} w$, we have the inequality

$$\|w\|_{H^{-s,r'}} \leq M|\mathrm{Re}(\zeta)|^{n(1/r-1/r')-2}\|P^* w\|_{H^{-s,r}}, \quad w \in \mathscr{S}. \tag{3.1}$$

Fix $f \in H^{s,r}$ and define a linear functional

$$l : P^*(\mathscr{S}) \subset H^{-s,r} \to \mathbb{C}, \quad l(P^* w) = (w, f)$$

where $(\,\cdot\,,\,\cdot\,)$ is the distributional pairing, defined to be conjugate linear in the second argument. By (3.1) any element of $P^*(\mathscr{S})$ has a unique representation as $P^* w$ for some $w \in \mathscr{S}$, so l is well defined and satisfies

$$\begin{aligned} |l(P^* w)| = |(w, f)| &\leq \|w\|_{H^{-s,r'}}\|f\|_{H^{s,r}} \\ &\leq M|\mathrm{Re}(\zeta)|^{n(1/r-1/r')-2}\|f\|_{H^{s,r}}\|P^* w\|_{H^{-s,r}}. \end{aligned}$$

By Hahn-Banach we may extend l as a continuous linear functional $\bar{l} : H^{-s,r} \to \mathbb{C}$ with the same norm bound, and by duality there is $v \in H^{s,r'}$ such that $\bar{l}(w) = (w, v)$ for $w \in H^{-s,r}$ and

$$\|v\|_{H^{s,r'}} \leq M|\mathrm{Re}(\zeta)|^{n(1/r-1/r')-2}\|f\|_{H^{s,r}}.$$

If $w \in \mathscr{S}$ we have

$$(w, Pv) = (P^* w, v) = \bar{l}(P^* w) = l(P^* w) = (w, f)$$

so that $Pv = f$.

By the above argument, for any $s \in \mathbb{R}$ there is a linear operator

$$G_\zeta : H^{s,r} \to H^{s,r'}, \quad f \mapsto v$$

where v solves $(-\Delta + 2\zeta \cdot D)v = f$, and one has the norm estimate

$$\|G_\zeta f\|_{H^{s,r'}} \leq M |\mathrm{Re}(\zeta)|^{n(1/r - 1/r') - 2} \|f\|_{H^{s,r}}.$$

This proves the result in the case $V = 0$.

Let us now consider nonzero V. Notice that one has

$$\|V G_\zeta f\|_{H^{s,r}} \leq AM |\mathrm{Re}(\zeta)|^{n(1/r - 1/r') - 2} \|f\|_{H^{s,r}}.$$

Since $n(1/r - 1/r') - 2 < 0$, we may choose $R > 0$ so that it satisfies $AMR^{n(1/r-1/r')-2} = 1/2$. Assuming that $|\mathrm{Re}(\zeta)| \geq R$, we have

$$\|V G_\zeta f\|_{H^{s,r}} \leq \frac{1}{2} \|f\|_{H^{s,r}}.$$

Now, we can solve $(-\Delta + 2\zeta \cdot D + V)u = f$ by taking $u = G_\zeta v$ where v is a solution of

$$(\mathrm{Id} + V G_\zeta) v = f.$$

By the above estimate, this equation for v can be solved by Neumann series and one has $\|v\|_{H^{s,r}} \leq 2\|f\|_{H^{s,r}}$. Then $u = G_\zeta v$ is the required solution and it satisfies

$$\begin{aligned} \|u\|_{H^{s,r'}} &\leq M |\mathrm{Re}(\zeta)|^{n(1/r - 1/r') - 2} \|v\|_{H^{s,r}} \\ &\leq 2M |\mathrm{Re}(\zeta)|^{n(1/r - 1/r') - 2} \|f\|_{H^{s,r}}. \end{aligned}$$ □

Proposition 3.4. *Let $n \in \{2, 3\}$, let $r = \frac{2(n+1)}{n+3}$ so that $r' = \frac{2(n+1)}{n-1}$, and assume that $V(x) = \chi_C(x)\varphi(x)$ where $C \subset \mathbb{R}^n$ is a closed circular cone with opening angle $< \pi$ and $\langle x \rangle^\alpha \varphi \in C^s(\mathbb{R}^n)$ for some $\alpha > 5/3$ and $0 < s < 1/2$ if $n = 2$ (respectively $\alpha > 9/4$ and $1/4 < s < 1/2$ if $n = 3$). Then $V \in H^{s-\varepsilon,r}(\mathbb{R}^n)$ for any $\varepsilon > 0$, and there is a constant $A > 0$ such that*

$$\|Vf\|_{H^{s-\varepsilon,r}} \leq A \|f\|_{H^{s-\varepsilon,r'}}.$$

Proof. We write $V = \tilde{\varphi} g$ where

$$\tilde{\varphi}(x) = \langle x \rangle^{\alpha} \varphi(x),$$
$$g(x) = \langle x \rangle^{-\alpha} \chi_C(x).$$

Also write $q = r'$ and $\tilde{q} = \frac{q}{q-2}$, so that

$$(q,\ q',\ \tilde{q}) = \begin{cases} (6,\ 6/5,\ 3/2) & \text{if } n = 2, \\ (4,\ 4/3,\ 2) & \text{if } n = 3. \end{cases}$$

We first observe that, by assumption,

$$\tilde{\varphi} \in C^s(\mathbb{R}^n).$$

Next, note that Proposition 3.7 gives that

$$g \in H^{\tau,p}(\mathbb{R}^n) \text{ if } 1 < p \leq 2,\ \alpha > n/p,\ \tau < 1/2.$$

The assumption on α implies

$$g \in H^{s,\tilde{q}} \cap H^{s,q'}(\mathbb{R}^n).$$

Since functions in $C^s(\mathbb{R}^n)$ act as pointwise multipliers on $H^{s-\varepsilon,p}(\mathbb{R}^n)$ for any $\varepsilon > 0$ and $1 < p < \infty$ [Tr92, Section 4.2], we have

$$V \in H^{s-\varepsilon,\tilde{q}} \cap H^{s-\varepsilon,q'}(\mathbb{R}^n).$$

Proposition 3.5 then implies that

$$\|Vf\|_{H^{s-\varepsilon,q'}} \leq A \|f\|_{H^{s-\varepsilon,q}}. \quad \square$$

Proposition 3.5. *Let $F \in H^{\tau,\widetilde{p}}(\mathbb{R}^n)$ where $\tau \in [0,1]$ and $\widetilde{p} = p/(p-2)$, and $p \geq 2$. Then the pointwise multiplier $T_F = f \longmapsto Ff$ maps*

$$T_F \colon H^{\tau,p}(\mathbb{R}^n) \longrightarrow H^{\tau,p'}(\mathbb{R}^n)$$

continuously.

For the proof of Proposition 3.5, we need the following well-known theorem on bilinear complex interpolation.

Theorem 3.6. *Let (A_0, A_1), (B_0, B_1), (C_0, C_1) be compatible Banach couples. Assume that*

$$T \colon (A_0 \cap A_1) \times (B_0 \cap B_1) \longrightarrow C_0 \cap C_1$$

is bilinear and one has the bounds

$$\|T(a,b)\|_{C_j} \le M_j \|a\|_{A_j} \|b\|_{B_j}$$

for $a \in A_0 \cap A_1, b \in B_0 \cap B_1, j = 0,1$. *Then the operator*

$$T\colon\ [A_0, A_1]_\vartheta \times [B_0, B_1]_\vartheta \longrightarrow [C_0, C_1]_\vartheta$$

is bounded for all $\vartheta \in (0,1)$ *with norm* $\le M_0^{1-\vartheta} M_1^\vartheta$, *where* $[\cdot,\cdot]_\vartheta$ *denotes the usual complex interpolation spaces.*

This is a special case of e.g. Theorem 4.4.1 in [BL76], see also the original theorem due to Calderón [Ca64].

Proof of Proposition 3.5. We first show that if $f \in L^p(\mathbb{R}^n)$ and $g \in L^{\widetilde{p}}(\mathbb{R}^n)$, where $p \ge 2$ and $\widetilde{p} = p/(p-2)$, then

$$fg \in L^{p'}(\mathbb{R}^n) \quad \text{and} \quad \big\|fg\big\|_{L^{p'}(\mathbb{R}^n)} \le \big\|f\big\|_{L^p(\mathbb{R}^n)} \big\|g\big\|_{L^{\widetilde{p}}(\mathbb{R}^n)}. \tag{$*$}$$

We use Hölder's inequality with $q = p/p' = p-1$ and $q' = (p-1)/(p-2)$, so that

$$\int_{\mathbb{R}^n} |fg|^{p'} \le \bigg(\int_{\mathbb{R}^n} |f|^{p'q} \bigg)^{1/q} \bigg(\int_{\mathbb{R}^n} |g|^{p'q'} \bigg)^{1/q'} .$$

Now $p'q = p$ and $p'q' = \widetilde{p}$ and $(*)$ follows.

Write $T(f,g) = fg$, so that $T_F(f) = T(f,F)$. We show that

a) $T\colon L^p(\mathbb{R}^n) \times L^{\widetilde{p}}(\mathbb{R}^n) \longrightarrow L^{p'}(\mathbb{R}^n)$ and

b) $T\colon H^{1,p}(\mathbb{R}^n) \times H^{1,\widetilde{p}}(\mathbb{R}^n) \longrightarrow H^{1,p'}(\mathbb{R}^n)$

continuously, and the the claim of Proposition 3.5 follows from Theorem 3.6. But a) is just $(*)$ and if $f \in H^{1,p}(\mathbb{R}^n)$ and $g \in H^{1,\widetilde{p}}(\mathbb{R}^n)$, then we have

$$\nabla(fg) = (\nabla f)g + f(\nabla g),$$

and b) follows from $(*)$. □

Proposition 3.7. *Let* $c > 0$ *and let* C *be the circular cone*

$$C = \big\{ (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,\big|\, |x'| \le c x_n \big\}.$$

Then $\langle x \rangle^{-\alpha} \chi_C(x)$ *belongs to* $H^{\tau,p}(\mathbb{R}^n)$ *if* $1 < p \le 2,\ \alpha > n/p,\ \tau < 1/2$.

Proposition 3.8. *Let $c > 0$ and let C be the circular cone*

$$C = \big\{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \;\big|\; |x'| \le c x_n \text{ and } x_n \le 1\big\}.$$

Then $\chi_C \in H^{\tau,p}(\mathbb{R}^n)$ for $\tau \in [0, 1/2)$ and $p \in (1, 2]$.

It is useful to first consider the easier case of characteristic functions of finite straight cylinders.

Proposition 3.9. *Let C' be the finite straight cylinder*

$$C' = \big\{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \;\big|\; |x'| \le 1 \text{ and } |x_n| \le 1\big\},$$

where $n \ge 2$. Then the characteristic function $\chi_{C'}$ belongs to $H^{\tau,p}(\mathbb{R}^n)$ for all $\tau \in [0, 1/2)$ and $p \in (1, 2]$.

Proof. The point is that we can compute the Fourier transform of $\chi_{C'}$ explicitly. First, since $\chi_{C'} \in L^1(\mathbb{R}^n)$, the Fourier transform $\widehat{\chi_{C'}}$ is continuous, and there are no singularities. Thus, only the decay of $\widehat{\chi_{C'}}$ needs to be considered.

Next, using the usual radial Fourier transform (see e.g. Sect. §IV.3 in [SW71]), and the fact that

$$\frac{d}{dx}(x^\nu\, J_\nu(x)) = x^\nu\, J_{\nu-1}(x),$$

(see e.g. [Le72], Section 5.2), we can compute the Fourier transform of the characteristic function of the ball $B = B^m(0,1) \subset \mathbb{R}^m$ for $m \ge 2$,

$$\begin{aligned}
\widehat{\chi_B}(\xi') &= \int_{|x'|<1} e^{-ix'\cdot\xi'}\, dx' = (2\pi)^{m/2} \int_0^1 J_{m/2-1}(|\xi'|\, s)\, s^{m/2}\, ds \\
&= (2\pi)^{m/2}\, |\xi'|^{-m/2} \int_0^1 (|\xi'|\, s)^{m/2}\, J_{m/2-1}(|\xi'|\, s)\, ds \\
&= (2\pi)^{m/2}\, |\xi'|^{-m/2}\, \left.\frac{(|\xi'|\, s)\, J_{m/2}(|\xi'|\, s)}{|\xi'|}\right]_0^{s=1} \\
&= (2\pi)^{m/2}\, |\xi'|^{-m/2}\, J_{m/2}(|\xi'|)
\end{aligned}$$

for all $\xi' \in \mathbb{R}^m \setminus 0$. By the asymptotics of J-Bessel functions (see e.g. [Le72], Section 5.11), this is $\lesssim \langle \xi' \rangle^{-m/2-1/2}$ for large $|\xi'|$. Similarly, for $\xi \in \mathbb{R} \setminus 0$, we get

$$\widehat{\chi_{[-1,1]}}(\xi) = \frac{2 \sin \xi}{\xi},$$

and this is $\lesssim \langle \xi \rangle^{-1}$ for large ξ.

Finally, since $\chi_{C'}(x', x_n) = \chi_B(x')\chi_{[-1,1]}(x_n)$ where B is the unit ball in $\mathbb{R}^{n-1}$, we have

$$|\widehat{\chi_{C'}}(\xi', \xi_n)| = |\widehat{\chi_B}(\xi')|\,|\widehat{\chi_{[-1,1]}}(\xi_n)| \lesssim \langle\xi'\rangle^{-n/2}\langle\xi_n\rangle^{-1}.$$

Thus we can estimate

$$\begin{aligned}\left\|\chi_{C'}\right\|^2_{H^{\tau,2}(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \langle\xi\rangle^{2\tau}\left|\widehat{\chi_{C'}}(\xi', \xi_n)\right|^2 d\xi'\,d\xi_n \\ &\lesssim \int_{\mathbb{R}^{n-1}} \langle\xi'\rangle^{2\tau}\langle\xi'\rangle^{-n}\,d\xi' \int_{\mathbb{R}} \langle\xi_n\rangle^{2\tau}\langle\xi_n\rangle^{-2}\,d\xi_n,\end{aligned}$$

and this product is finite for any $\tau < 1/2$, so $\chi_{C'} \in H^{\tau,2}$ for $\tau < 1/2$.

More generally, choose a fixed $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $\chi_{C'} = \varphi\chi_{C'}$, and note that for $1 < p \le 2$ we have by the Hölder inequality

$$\|\varphi f\|_{H^{k,p}} \le C_\varphi \|f\|_{H^{k,2}}, \qquad k = 0, 1.$$

By interpolation, multiplication by φ maps $H^{s,2}$ to $H^{s,p}$ for $0 < s < 1$, so we have $\chi_{C'} \in H^{\tau,p}$ for $\tau < 1/2$ and $1 < p \le 2$. □

Proof of Proposition 3.8. So, let $\tau < 1/2$. Through a simple change of variables, it is enough to consider the cone

$$C = \left\{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \;\middle|\; |x'| \le x_n \le 1\right\}.$$

We shall break this cone into pieces of the form

$$C(\alpha, \beta) = \left\{(x', x_n) \in C \;\middle|\; \alpha \le x_n \le \beta\right\},$$

where $\alpha, \beta \in \mathbb{R}_+$. By a simple change of variables, Prop. 3.9 shows that $\chi_{C(1/2,1)} \in H^{\tau,p}(\mathbb{R}^n)$. Since

$$\chi_{C(1/4,1/2)} = \chi_{C(1/2,1)}(2\cdot), \quad \chi_{C(1/8,1/4)} = \chi_{C(1/2,1)}(4\cdot), \quad \ldots,$$

we also have

$$\chi_{C(1/2,1)}, \chi_{C(1/4,1/2)}, \chi_{C(1/8,1/4)}, \ldots \in H^{\tau,p}(\mathbb{R}^n),$$

and

$$\chi_C = \sum_{j=1}^{\infty} \chi_{C(2^{-j},2^{1-j})},$$

in the sense of distributions, and so it suffices to prove that this series converges in $H^{\tau,p}(\mathbb{R}^n)$. Finally, this follows from the scaling of the Sobolev norm, see Proposition 3.10 below. □

Proposition 3.10. *Let $\tau \in [0,1]$, $p \in (1,\infty)$, and let $\psi \in H^{\tau,p}(\mathbb{R}^n)$. Then, for $\lambda \in [1,\infty)$,*

$$\|\psi(\lambda\cdot)\|_{H^{\tau,p}(\mathbb{R}^n)} \lesssim \lambda^{\tau-n/p}\|\psi\|_{H^{\tau,p}(\mathbb{R}^n)},$$

and for $\lambda \in (0,1]$,

$$\|\psi(\lambda\cdot)\|_{H^{\tau,p}(\mathbb{R}^n)} \lesssim \lambda^{-n/p}\|\psi\|_{H^{\tau,p}(\mathbb{R}^n)}.$$

Proof. The result follows from complex interpolation once the cases $\tau = 0$ and $\tau = 1$ have been dealt with. The case $\tau = 0$ follows immediately from

$$\|\psi(\lambda\cdot)\|_{L^p(\mathbb{R}^n)} = \lambda^{-n/p}\|\psi\|_{L^p(\mathbb{R}^n)}.$$

When $\tau = 1$, we have

$$\begin{aligned}
\|\psi(\lambda\cdot)\|_{H^{1,p}(\mathbb{R}^n)} &\lesssim \|\psi(\lambda\cdot)\|_{L^p(\mathbb{R}^n)} + \sum_{j=1}^{n}\|\partial_j(\psi(\lambda\cdot))\|_{L^p(\mathbb{R}^n)} \\
&= \lambda^{-n/p}\|\psi\|_{L^p(\mathbb{R}^n)} + \lambda\sum_{j=1}^{n}\|(\partial_j\psi)(\lambda\cdot)\|_{L^p(\mathbb{R}^n)} \\
&\leq \max\left\{\lambda^{-n/p}, \lambda^{1-n/p}\right\}\|\psi\|_{H^{1,p}(\mathbb{R}^n)}.
\end{aligned}$$

□

Proof of Proposition 3.7. We continue to use the notation $C(\alpha,\beta)$ from the proof of Proposition 3.8. Proposition 3.8 tells us that $\chi_{C(0,1)} \in H^{\tau,p}(\mathbb{R}^n)$, and since $\langle\cdot\rangle^{-\alpha} \in C^\infty(\mathbb{R}^n)$, also $\langle\cdot\rangle^{-\alpha}\chi_{C(0,1)} \in H^{\tau,p}(\mathbb{R}^n)$.

Thus, it suffices to prove that $\langle\cdot\rangle^{-\alpha}\chi_{C(1,\infty)} \in H^{\tau,p}(\mathbb{R}^n)$. We split this into series

$$\sum_{j=0}^{\infty}\langle\cdot\rangle^{-\alpha}\chi_{C(2^j,2^{j+1})} = \sum_{j=0}^{\infty}\langle\cdot\rangle^{-\alpha}\chi_{C(1,2)}\left(\frac{\cdot}{2^j}\right),$$

and each of the individual terms belongs to $H^{\tau,p}(\mathbb{R}^n)$, and so it only remains to prove that the series converges in $H^{\tau,p}(\mathbb{R}^n)$. For this purpose, let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be such that $\varphi \equiv 1$ on $\operatorname{supp}\chi_{C(1,2)}$ and such that φ is supported on an ε-neighbourhood of $\operatorname{supp}\chi_{C(1,2)}$ for some $\varepsilon \in (0,1/4)$, say. Now we may estimate

$$\begin{aligned}
&\|\langle\cdot\rangle^{-\alpha}\chi_{C(1,2)}(\cdot/2^j)\|_{H^{\tau,p}(\mathbb{R}^n)} = \|\langle\cdot\rangle^{-\alpha}\varphi(\cdot/2^j)\chi_{C(1,2)}(\cdot/2^j)\|_{H^{\tau,p}(\mathbb{R}^n)} \\
&\lesssim \|\langle\cdot\rangle^{-\alpha}\varphi(\cdot/2^j)\|_{C^1(\mathbb{R}^n)}\|\chi_{C(1,2)}(\cdot/2^j)\|_{H^{\tau,p}(\mathbb{R}^n)} \lesssim 2^{-\alpha j}\cdot 2^{jn/p},
\end{aligned}$$

and since the exponent $-\alpha+n/p$ is negative, the series in question is comparable to geometric series and converges, as required. □

Proof of Theorem 3.1. We look for a solution of $(-\Delta+V-\lambda)u=0$ of the form $u=e^{i\zeta\cdot x}(1+\psi)$ where $\zeta\in\mathbb{C}^n$ satisfies $\zeta\cdot\zeta=\lambda$. Now u will be a solution if and only if ψ satisfies

$$(-\Delta+2\zeta\cdot D+V)\psi=-V.$$

We wish to use Propositions 3.3 and 3.4 to solve this equation. Note that Proposition 3.4 has the assumption $r=\frac{2(n+1)}{n+3}$, which implies that

$$\frac{1}{r}-\frac{1}{r'}=\frac{2}{n+1}.$$

This is consistent with Proposition 3.3, which requires that

$$\frac{1}{r}-\frac{1}{r'}\in\left[\frac{2}{n+1},\frac{2}{n}\right).$$

We have chosen r so that $1/r-1/r'$ is as small as possible, in order to arrange the best power of $|\mathrm{Re}(\zeta)|$ in the estimates.

Note that one has

$$(r,\ r',\ 1/r-1/r')=\begin{cases}(6/5,\ 6,\ 2/3) & \text{if } n=2,\\ (4/3,\ 4,\ 1/2) & \text{if } n=3.\end{cases}$$

By Proposition 3.4, the function V satisfies the condition in Proposition 3.3 with s replaced by some $t<s$, where $t>0$ if $n=2$ and $t>1/4$ if $n=3$. If $|\mathrm{Re}(\zeta)|$ is sufficiently large, we obtain a solution ψ satisfying

$$\|\psi\|_{H^{t,r'}}\le C|\mathrm{Re}(\zeta)|^{n(1/r-1/r')-2}\|V\|_{H^{t,r}}\le C|\mathrm{Re}(\zeta)|^{-2/(n+1)}.$$

We have the Sobolev embedding $H^{t,r'}\subset L^q$ where $q=\frac{nr'}{n-tr'}>r'$ (here we assume $t<n/r'$). Thus we have

$$\|\psi\|_{L^q}\le C|\mathrm{Re}(\zeta)|^{-\frac{2}{n+1}}=C|\mathrm{Re}(\zeta)|^{-\frac{n}{q}-\delta}$$

where $\delta=\frac{2}{n+1}-\frac{n}{q}=\frac{2}{n+1}-\frac{n-tr'}{r'}>0$ by our assumptions on r' and t. The result follows by taking $\rho=-i\zeta$. $\square$

4 Reduction to Laplace transform

We also need to analyze further the solution v_0 in Proposition 2.1. As a solution of $(-\Delta-\lambda)v_0=0$ in $\mathbb{R}^n$, v_0 is real-analytic and has a Taylor series at the origin. If v_0 is nonzero, the Taylor series is not identically zero. Assume that the first nonvanishing term has degree N, and denote by $H(x)$ the homogeneous polynomial of all terms of degree N. Thus

$$v_0(x) = H(x) + R(x), \qquad |R(x)| \leq C|x|^{N+1},$$

for x near the origin. The next observation is [BPS14, Lemma 3.4].

Lemma 4.1. *If $v_0 \in B^*$ is a nontrivial solution of $(-\Delta - \lambda)v_0 = 0$, then the lowest degree nonvanishing terms H in the Taylor expansion of v_0 form a harmonic homogeneous polynomial in $\mathbb{R}^n$.*

The main point in this section is the following result, proved by using the solutions of Theorem 3.1 in Proposition 2.1. This result implies that whenever λ is a non-scattering energy, then the Laplace transform of $\chi_C H$ vanishes in a certain complex manifold.

Proposition 4.2. *Let $\lambda > 0$, let $n \in \{2, 3\}$, and let $V(x) = \chi_C(x)\varphi(x)$ where $\varphi \in \langle\cdot\rangle^{-\alpha} C^s(\mathbb{R}^n)$ is superexponentially decaying with $\alpha > 5/3$ and $s > 0$ for $n = 2$ (resp. $\alpha > 9/4$ and $s > 1/4$ for $n = 3$), $\varphi(0) \neq 0$, and $C \subset \mathbb{R}^n$ is a closed circular cone opening in direction e_n with angle $< \pi$ having vertex at the origin. Let U be a neighborhood of e_n in S^{n-1} such that $e^{-\tau\omega\cdot x}$ is exponentially decaying in C whenever $\tau > 0$ and $\omega \in \overline{U}$.*

Assume that λ is a non-scattering energy for V, let v_0 be the solution in Proposition 2.1, and write $v_0 = H + R$ where H is a harmonic homogeneous polynomial of degree N as in Lemma 4.1. Then

$$\int_C e^{-\rho\cdot x} H(x)\, dx = 0, \qquad \rho \in W_0,$$

where W_λ is the set of all $\rho \in \mathbb{C}^n$ of the form

$$\rho = \rho(\tau, \omega, \omega') = \tau\omega + i(\tau^2 + \lambda)^{1/2}\omega'$$

and where the parameters satisfy

$$\tau > M, \quad \omega \in U, \quad \omega' \in S^{n-1} \text{ with } \omega\cdot\omega' = 0$$

for some sufficiently large M.

Notation: All implicit constants below are allowed to depend on all the parameters except for ρ.

Proof. Let $\rho \in W_\lambda$. Theorem 3.1 guarantees the existence of a CGO solution $u = e^{-\rho \cdot x}(1+\psi)$, where ψ depends on ρ, with

$$\|\psi\|_{L^q(\mathbb{R}^n)} \lesssim \frac{1}{|\rho|^c},$$

where $q = 2(n+1)/(n-1)$ and $c = n(n-1)/2(n+1) + \delta$ for some small $\delta \in \mathbb{R}_+$. To simplify notation, we assume, as we may, that $\delta < s$. We also remark that $(V(x)-1)/(|x|^s)$ is bounded in C, where for simplicity we have assumed that $\varphi(0) = 1$.

We define $F(\rho)$ to be the Laplace transform

$$F(\rho) := \int_C e^{-\rho \cdot x} H(x)\, dx, \qquad \mathrm{Re}(\rho) \in U.$$

Our goal is to prove that

$$F(\rho) = 0, \qquad \rho \in W_0. \tag{4.1}$$

To do this, we observe that Proposition 2.1 yields

$$\int_{\mathbb{R}^n} V e^{-\rho \cdot x}(1+\psi)(H+R)\, dx = 0, \qquad \rho \in W_\lambda,$$

which can be rewritten as

$$F(\rho) = -\int_C e^{-\rho \cdot x}\big[(V-1)H + V(R + \psi(H+R))\big]\, dx, \quad \rho \in W_\lambda. \tag{4.2}$$

By the homogeneity of $H(x)$ and using that C is a cone, the left hand side satisfies

$$F(\rho) = |\rho|^{-N-n} F(\rho/|\rho|).$$

For the right hand side, we claim that for all $\rho \in W_\lambda$ one has

$$\left| \int_C e^{-\rho \cdot x}\big[(V-1)H + V(R + \psi(H+R))\big]\, dx \right| \lesssim |\rho|^{-N-n-\delta}. \tag{4.3}$$

Assuming that (4.3) holds, (4.2) implies

$$F\left(\frac{\rho}{|\rho|}\right) \lesssim |\rho|^{-\delta},$$

which holds for $\rho \in W_\lambda$, and more precisely,

$$F\left(\frac{\tau\omega}{\sqrt{2\tau^2+\lambda}} + \frac{i\omega'\sqrt{\tau^2+\lambda}}{\sqrt{2\tau^2+\lambda}}\right) \lesssim \left(\sqrt{2\tau^2+\lambda}\right)^{-\delta} \lesssim \tau^{-\delta}$$

for $\tau \gg 1$, $\omega \in U$ and $\omega' \in S^{n-1}$ with $\omega' \perp \omega$. Now, taking $\tau \longrightarrow \infty$ gives

$$F\left(\frac{\omega}{\sqrt{2}} + \frac{i\omega'}{\sqrt{2}}\right) = 0,$$

which holds for all $\omega \in U$ and all $\omega' \in S^{n-1}$ with $\omega \perp \omega'$. By homogeneity we have

$$F(t\omega + it\omega') = 0$$

for all $t \in \mathbb{R}_+$, $\omega \in U$ and $\omega' \in S^{n-1}$ with $\omega \perp \omega'$, which proves (4.1) as required.

It remains to show (4.3). We shall split the left hand side of (4.3) into many integrals which are easier to estimate. The following is essentially Lemma 3.6 from [BPS14].

Lemma 4.3. *Let $R\colon \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{C}$ be a continuous homogeneous function of degree N, and let $e^{-\mathrm{Re}(\rho)\cdot x}$ be exponentially decaying in C. Then, for any $f \in L^q(\mathbb{R}^n)$, where $q \in [1, \infty)$, we have*

$$\int_C e^{-\rho\cdot x} R(x) f(x)\, dx \lesssim |\rho|^{n/q-N-n} \left\| e^{-(\rho/|\rho|)\cdot x} R \right\|_{L^{q'}(\mathbb{R}^n)} \left\| f \right\|_{L^q(\mathbb{R}^n)}.$$

This follows immediately from the change of variables $y = x/|\rho|$ and Hölder's inequality.

First we observe that, by Hölder's inequality,

$$\begin{aligned}\int_{C\setminus B(0,\varepsilon)} & e^{-\rho\cdot x} \left(1 + \psi(x)\right) V(x)\, v_0(x)\, dx \\ & \lesssim e^{-\varepsilon d|\rho|} \left(\left\| V v_0 \right\|_{L^1(\mathbb{R}^n)} + \left\| \psi \right\|_{L^q(\mathbb{R}^n)} \left\| V v_0 \right\|_{L^{q'}(\mathbb{R}^n)} \right) \lesssim e^{-\varepsilon d|\rho|},\end{aligned}$$

where d is some suitably small positive real constant. We also have

$$\int_{C\setminus B(0,\varepsilon)} e^{-\rho\cdot x}\, H(x)\, dx \lesssim e^{-\varepsilon d|\rho|}.$$

Also, Lemma 4.3 gives, observing that $n(n-1)/2(n+1) - c = -\delta$, the estimates

$$\int_{C\cap B(0,\varepsilon)} e^{-\rho\cdot x} H(x)\,(V(x)-1)\;dx$$
$$= \int_{C\cap B(0,\varepsilon)} e^{-\rho\cdot x} H(x)\,|x|^s\,\frac{V(x)-1}{|x|^s}\,dx \lesssim |\rho|^{-N-n-s}\,,$$
$$\int_{C\cap B(0,\varepsilon)} e^{-\rho\cdot x}\,\psi(x)\,V(x)\,H(x)\,dx \lesssim |\rho|^{-N-n-\delta}\,,$$
$$\int_{C\cap B(0,\varepsilon)} e^{-\rho\cdot x}\,\psi(x)\,V(x)\,O(|x|^{N+1})\,dx \lesssim |\rho|^{-N-n-1-\delta}\,, \quad \text{and}$$
$$\int_{C\cap B(0,\varepsilon)} e^{-\rho\cdot x}\,V(x)\,O(|x|^{N+1})\,dx \lesssim |\rho|^{-N-n-1}\,.$$

Combining the above estimates gives the desired claim (4.3). □

5 The two-dimensional case

We now restrict our attention to the two-dimensional case and prove Theorem 1.1. Assume for the sake of contradiction that $\lambda > 0$ is a non-scattering energy for a potential V as in Theorem 1.1. By Proposition 4.2, this implies the vanishing of the following Laplace transform for a nonzero homogeneous harmonic polynomial H of degree N,

$$\int_C e^{-(\omega+i\omega')\cdot x}\,H(x)\,dx = 0$$

for all $\omega \in U$ where U is an open subset of the unit circle S^1, and for all $\omega' \in S^1$ is such that $\omega \perp \omega'$. Since we are in two dimensions, H must be of the form

$$H(x) = a\,(x_1+ix_2)^N + b\,(x_1-ix_2)^N$$

for some constants a and b, when $N > 0$. If $N = 0$, then $H(x)$ will be just a constant a. The goal is to prove that $a = b = 0$, which will contradict the fact that H is nonzero and will prove Theorem 1.1.

First we introduce some notation. For simplicity, we assume that the cone C opens in direction e_1 instead of e_2 as in Proposition 4.2. We shall write S for the arc $S^1 \cap C$, and I for the interval $[-L/2, L/2]$ parametrizing S under the mapping $r \longmapsto e^{ir}\colon\ (-\pi,\pi] \longrightarrow S^1$. Here $L \in (0,\pi)$ is the opening angle of the sector C. Now we can take U to be the arc of S^1 corresponding, in the same coordinates, to the interval $(-\pi/2 + L/2, \pi/2 - L/2)$.

In polar coordinates, we have

$$\int_S \int_0^\infty e^{-(\omega+i\omega')\cdot\vartheta\, r}\, r^{N+1}\, dr\, H(\vartheta)\, d\vartheta = 0.$$

We wish to rewrite the r-integral. Let $\alpha \in \mathbb{C}$ have a positive real part (we will take $\alpha = (\omega + i\omega')\cdot\vartheta$). Then, by Cauchy's integral theorem, as the integrand is exponentially decaying in the right half-plane of the complex plane, we can rotate the path of integration from the half-ray $\{\alpha r \,|\, r \in \mathbb{R}_+\}$ to $\mathbb{R}_+$ giving

$$\int_0^\infty e^{-\alpha r}\,(\alpha r)^{N+1}\,\alpha\, dr = \int_0^\infty e^{-r}\, r^{N+1}\, dr = \Gamma(N+2) = (N+1)!.$$

Thus we obtain

$$\int_S ((\omega + i\omega')\cdot\vartheta)^{-N-2}\, H(\vartheta)\, d\vartheta = 0.$$

With the parametrization $\omega = e^{i\varphi}$ and $\vartheta = e^{i\psi}$, we can compute

$$\begin{aligned}(\omega + i\omega')\cdot\vartheta &= \begin{bmatrix}\cos\varphi \mp i\sin\varphi\\ \sin\varphi \pm i\cos\varphi\end{bmatrix}\cdot\begin{bmatrix}\cos\psi\\ \sin\psi\end{bmatrix} = \begin{bmatrix}e^{\mp i\varphi}\\ \pm ie^{\mp i\varphi}\end{bmatrix}\cdot\begin{bmatrix}\cos\psi\\ \sin\psi\end{bmatrix}\\ &= e^{\mp i\varphi}(\cos\psi \pm i\sin\psi) = e^{\mp i\varphi}e^{\pm i\psi} = e^{\mp i(\varphi-\psi)} = e^{\pm i(\psi-\varphi)}.\end{aligned}$$

Thus, the expression involving ω and ω' factors nicely and the variables φ and ψ become separated.

In the case $N = 0$ we then have

$$a\int_S e^{\mp i2\psi} dx = 0,$$

or more simply

$$a\,\widehat{\chi_I}(\pm 2) = 0.$$

The Fourier coefficient is easy to compute and we get

$$a\sin L = 0,$$

and so we must have $H(x) \equiv a = 0$.

When $N > 0$ we get

$$\int_S e^{\mp i(N+2)\psi}\left(ae^{iN\psi} + be^{-iN\psi}\right) d\psi = 0.$$

This leads to the pair of equations

$$\begin{cases} a\int_{-L/2}^{L/2} e^{-2i\psi}\, d\psi & +b\int_{-L/2}^{L/2} e^{-i(2N+2)\psi}\, d\psi & = 0, \\ a\int_{-L/2}^{L/2} e^{i(2N+2)\psi}\, d\psi & +b\int_{-L/2}^{L/2} e^{2i\psi}\, d\psi & = 0. \end{cases}$$

In terms of Fourier coefficients this reads

$$\begin{cases} a\,\widehat{\chi_I}(2) & +b\,\widehat{\chi_I}(2N+2) & = 0, \\ a\,\widehat{\chi_I}(2N+2) & +b\,\widehat{\chi_I}(2) & = 0, \end{cases}$$

where we have used the fact that χ_I is even. This is a homogeneous linear system of equations for a and b, and if the determinant of the coefficient matrix is nonzero, then we must have $a = b = 0$. The determinant can vanish only if

$$\widehat{\chi_I}(2) = \pm\widehat{\chi_I}(2N+2).$$

The Fourier coefficients of χ_I are easy to compute, and the vanishing of the determinant simplifies to

$$\sin L = \pm\frac{1}{N+1}\sin\left((N+1)L\right).$$

It is now straightforward to check that this equation has no solutions L in the interval $(0,\pi)$. The derivative

$$\frac{d}{dL}\left(\sin L \mp \frac{1}{N+1}\sin\left((N+1)L\right)\right) = \cos L \mp \cos\left((N+1)L\right)$$

is clearly positive when $0 < L < \pi/(N+1)$. When

$$L \in \left[\pi/(N+1), N\pi/(N+1)\right],$$

we clearly have

$$\sin L > \frac{1}{N+1} \geq \frac{1}{N+1}\sin\left((N+1)L\right).$$

Finally, the case where L belongs to $(N\pi/(N+1),\pi)$ reduces to the case where L belongs to $(0,\pi/(N+1))$ by the change of variables $L \longmapsto \pi - L$.

6 The three-dimensional case

By the same argument as in the beginning of Section 5, the proof of Theorem 1.2 reduces to showing the following result.

Lemma 6.1. *Let $n = 3$, and let $S_\gamma = \{x \in S^{n-1}\,;\, x_n > \cos\gamma\}$ be a spherical cap where $0 < \gamma < \pi/2$. There is a countable subset $E \subset (0, \pi/2)$ such that for any $\gamma \in (0, \pi/2) \setminus E$, the condition*

$$\int_{S_\gamma} ((e_n + i\eta) \cdot x)^{-N-n} H(x)\, dx = 0, \qquad \eta \in S^{n-1}, \ \ \eta \cdot e_n = 0,$$

implies that $H \equiv 0$ whenever H is a spherical harmonic on S^{n-1} of degree N.

To prepare for the proof, write $x = ((\sin\alpha)\omega', \cos\alpha)$ where $\omega' \in S^{n-2}$. Writing also $\eta = (\eta', 0)$ where $\eta' \in S^{n-2}$, the integral becomes

$$\begin{aligned} &\int_{S_\gamma} ((e_n + i\eta) \cdot x)^{-N-n} H(x)\, dx \\ &= \int_0^\gamma \int_{S^{n-2}} (\cos\alpha + i(\sin\alpha)\eta' \cdot \omega')^{-N-n} H((\sin\alpha)\omega', \cos\alpha) \sin^{n-2}\alpha\, d\omega'\, d\alpha. \end{aligned}$$

Let $\{Y_1^N, \ldots, Y_r^N\}$ be some basis of spherical harmonics of degree N where $r = r_N$, and write $H = \sum_{j=1}^r a_j Y_j^N$. It is convenient to rephrase this in terms of rotation matrices: we write $\eta' = Re_1$ where R is a rotation matrix, that is, $R \in SO(n-1)$. The condition in Lemma 6.1 then becomes

$$\sum_{j=1}^r a_j f_j^N(\gamma; R) = 0, \qquad R \in SO(n-1), \tag{6.1}$$

where

$$\begin{aligned} &f_j^N(\gamma; R) := \\ &\quad \int_0^\gamma \int_{S^{n-2}} (\cos\alpha + i(\sin\alpha)\omega_1')^{-N-n} Y_j^N((\sin\alpha)R\omega', \cos\alpha) \sin^{n-2}\alpha\, d\omega'\, d\alpha. \end{aligned}$$

Here we have changed variables $\omega' \mapsto R\omega'$ in the integral (note that this uses the fact that S_γ is a spherical cap).

The next result together with an analyticity argument will imply Lemma 6.1.

Lemma 6.2. *Assume that $n = 3$. For any $N \geq 0$, there exists a basis $\{Y_1^N, \ldots, Y_r^N\}$ of spherical harmonics of degree N and there exist rotation matrices $R_1, \ldots, R_r \in SO(n-1)$ such that the function*

$$g^N : (0, \pi/2) \to \mathbb{C}, \ \ g^N(\gamma) := \det\left[(f_j(\gamma; R_k))_{j,k=1}^r\right]$$

is not identically zero.

Proof. Assume that $n = 3$. In this case there is an explicit basis of spherical harmonics of degree N given by

$$Y_j^N((\sin\alpha)\omega', \cos\alpha) = P_N^{|j|}(\cos\alpha)e^{ij\beta}, \quad -N \leq j \leq N$$

where P_N^m are associated Legendre polynomials and $\omega' = (\cos\beta, \sin\beta)$. (As is customary, we index the basis by $-N \leq j \leq N$ instead of $1 \leq j \leq 2N+1$.) Let R_k be the rotation in S^1 by angle θ_k. Then

$$Y_j^N((\sin\alpha)R_k\omega', \cos\alpha) = e^{ij\theta_k}Y_j^N((\sin\alpha)\omega', \cos\alpha).$$

This implies that

$$f_j^N(\gamma; R_k) = e^{ij\theta_k}f_j(\gamma)$$

where $f_j(\gamma) := f_j^N(\gamma; \mathrm{Id})$, and

$$g^N(\gamma) = f_1(\gamma)\cdots f_r(\gamma)\det\left[(e^{ij\theta_k})_{j,k=-N}^N\right].$$

The last determinant is of Vandermonde type. We choose the rotations so that $e^{i\theta_k} \neq e^{i\theta_l}$ for $k \neq l$, and then the last determinant is nonzero.

To show that $g^N(\gamma)$ is not identically zero, we need to demonstrate that there is some $\gamma \in (0, \pi/2)$ such that the product $f_1(\gamma)\cdots f_r(\gamma)$ is nonzero. We first prove that none of the functions f_j is identically zero in $(0, \pi/2)$. Now

$$f_j(\gamma) = \int_0^\gamma \int_{S^1} (\cos\alpha + i(\sin\alpha)\omega_1')^{-N-n}Y_j^N((\sin\alpha)\omega', \cos\alpha)\sin\alpha\, d\omega'\, d\alpha.$$

Each f_j extends analytically near $[0, \pi/2)$, and its derivative satisfies

$$f_j'(\gamma) = \sin\gamma\int_{S^1}(\cos\gamma + i(\sin\gamma)\omega_1')^{-N-n}Y_j^N((\sin\gamma)\omega', \cos\gamma)\, d\omega'.$$

Inserting the explicit form for Y_j^N we get

$$f_j'(\gamma) = P_N^{|j|}(\cos\gamma)\sin\gamma\int_0^{2\pi}(\cos\gamma + i\sin\gamma\cos\beta)^{-N-n}e^{ij\beta}\, d\beta.$$

It is enough to show that the function $\gamma \mapsto \int_0^{2\pi}\cdots\, d\beta$ is not identically zero. For $j = 0$ this follows just by taking $\gamma = 0$, and for $j \neq 0$ the result follows by differentiating $|j|$ times with respect to γ and taking $\gamma = 0$. More precisely, writing $p = \cos\gamma + i\sin\gamma\cos\beta$ and $p' = dp/d\gamma = -\sin\gamma + i\cos\gamma\cos\beta$, the $|j|$th derivative of the integral has the form

$$\int_0^{2\pi} \frac{\nu_0 (p')^{|j|} + \nu_1 p (p')^{|j|-1} + \ldots + \nu_{|j|} p^{|j|}}{p^{N+n+|j|}} e^{ij\beta}\, d\beta,$$

for some constants $\nu_0, \nu_1, \ldots, \nu_{|j|} \in \mathbb{C}$, and in particular, the coefficient ν_0 is

$$\nu_0 = \pm (N+n)(N+n+1)\cdots(N+n+|j|-1) \neq 0.$$

At $\gamma = 0$, we have $p = 1$ and $p' = i\cos\beta$, and the integral simplifies to

$$\int_0^{2\pi} \left(\nu_0' \cos^{|j|}\beta + \nu_1' \cos^{|j|-1}\beta + \ldots + \nu_{|j|}' \right) e^{ij\beta}\, d\beta,$$

where the coefficients ν_0', ν_1', ... are the same coefficients as before except for the obvious powers of i. Writing the cosines in terms of exponentials, there will be exactly one term which resonates with $e^{ij\beta}$, namely the exponential $e^{-ij\beta}$ coming from $\cos^{|j|}\beta$, and its coefficient is nonzero. Thus the $|j|$th derivative of $\int_0^{2\pi} \ldots d\beta$ at $\gamma = 0$ is nonzero, and as an analytic function of γ, the integral can not be identically zero.

We have proved that each f_j is not identically zero, and since f_j extends analytically near $[0,\pi/2)$ it is nonvanishing in $(0,\pi/2)\setminus E_j$ for some countable discrete subset $E_j \subset (0,\pi/2)$. Then $f_1 \cdots f_r$ is nonvanishing in $(0,\pi/2)\setminus E$ where $E = \cup_{j=1}^r E_j$ is a countable set. □

Proof of Lemma 6.1. Each function $\gamma \mapsto f_j^N(\gamma, R)$ extends as an analytic function in some neighborhood of the interval $[0,\pi/2)$ in the complex plane, and the same is true for the functions g^N in Lemma 6.2. For each N, by Lemma 6.2 we can choose Y_j^N and R_j such that g^N is analytic in some neighborhood U_N of $[0,\pi/2)$ and $g^N|_{(0,\pi/2)}$ is not identically zero. By analyticity the set $E_N = \{z \in U_N\,;\, g^N(z) = 0\}$ is countable and discrete in U_N.

Define

$$E = \bigcup_{N=0}^{\infty} (E_N \cap (0,\pi/2)).$$

Then E is a countable subset of $(0,\pi/2)$, and each g^N is nonvanishing in $(0,\pi/2)\setminus E$.

Assume now that $\gamma \in (0,\pi/2)\setminus E$, let $N \geq 0$, and let H be a spherical harmonic of degree N such that the condition in Lemma 6.1 holds. Writing

$H = \sum_{j=1}^r a_j Y_j^N$ where Y_j^N and R_j were chosen above, by (6.1) we have

$$\sum_{j=1}^r a_j f_j^N(\gamma; R_k) = 0, \qquad k = 1, \dots, r.$$

But $g^N(\gamma) \neq 0$ so the matrix $(f_j^N(\gamma; R_k))_{j,k=1}^r$ is invertible, which implies that $a_j = 0$ for $j = 1, \dots, r$. This proves that $H \equiv 0$. □

References

[BL76] J. Bergh, J. Löfström, *Interpolation Spaces. An Introduction*, Grundlehren der mathematischen Wissenschaften, **223**, Springer–Verlag, 1976.

[BPS14] E. Blåsten, L. Päivärinta, J. Sylvester, *Corners always scatter*, Comm. Math. Phys. (to appear).

[CGH10] F. Cakoni, D. Gintides, H. Haddar, *The existence of an infinite discrete set of transmission eigenvalues*, SIAM Journal on Mathematical Analysis **42** (2010), 237–255.

[CH13a] F. Cakoni, H. Haddar, *Transmission eigenvalues in inverse scattering theory*, in [Uh13], 529–578.

[CH13b] F. Cakoni, H. Haddar, *Transmission eigenvalues*, Inverse Problems **29** (2013), 100201, 1–3.

[Ca64] A.P. Calderón, *Intermediate spaces and interpolation, the complex method*, Studia Math. **24** (1964), 113–190.

[Ca80] A.P. Calderón, *On an inverse boundary value problem*, Seminar on Numerical Analysis and its Applications to Continuum Physics, Soc. Brasileira de Matemática, Río de Janeiro, 1980.

[CK96] D. Colton, A. Kirsch, *A simple method for solving inverse scattering problems in the resonance region*, Inverse Problems **12** (1996), 383– 393.

[CM88] D. Colton, P. Monk, *The inverse scattering problem for time-harmonic acoustic waves in an inhomogeneous medium*, The Quarterly Journal of Mechanics and Applied Mathematics **41** (1988), 97–125.

[GH12] J. Gell-Redman, A. Hassell, *Potential scattering and the continuity of phase-shifts*, Math. Res. Lett. **19** (2012), 719–729.

[GM86] P.G. Grinevich, S.V. Manakov: *The inverse scattering problem for the two-dimensional Schrödinger operator, the $\overline{\partial}$-method and non-linear equations*, Functional Analysis and its Applications **20** (1986), 94–103.

[GN95] P. Grinevich, R.G. Novikov, *Transparent potentials at fixed energy in dimension two. Fixed-energy dispersion relations for the fast decaying potentials*, Comm. Math. Phys. **174** (1995), 409-446.

[Hö83] L. Hörmander, *The analysis of linear partial differential operators*, vol. I-II, Springer-Verlag, Berlin, 1983.

[KRS87] C. E. Kenig, A. Ruiz, C.D. Sogge, *Uniform Sobolev inequalities and unique continuation for second order constant coefficient differential operators*, Duke Math. J. **55** (1987), 329–347.

[Ki86] A. Kirsch, *The denseness of the far field patterns for the transmission problem*, IMA Journal of Applied Mathematics **37** (1986), 213–225.

[KN08] A. Kirsch, N. Grinberg, The factorization method for inverse problems. Oxford Lecture Series in Mathematics and its Applications **36**. Oxford University Press, Oxford, 2008.

[Le72] N.N. Lebedev, *Special Functions & Their Applications*, Dover Publications, 1972.

[Me95] R.B. Melrose, *Geometric scattering theory*, Cambridge University Press, 1995.

[Na88] A. Nachman, *Reconstructions from boundary measurements*, Ann. of Math. **128** (1988), 531-576.

[Ne62] R.G. Newton, *Construction of potentials from the phase shifts at fixed energy*, J. Mathematical Phys. **3** (1962), 75–82.

[No88] R.G. Novikov, *A multidimensional inverse spectral problem for the equation* $-\Delta\psi + (v(x) - Eu(x))\psi = 0$, Funct. Anal. Appl. **22** (1988), 263–272.

[No08] R.G. Novikov, *The* $\bar{\partial}$*–approach to monochromatic inverse scattering in three dimensions*, J. Geom. Anal. **18** (2008), 612–631.

[NK87] R.G. Novikov, G.M. Khenkin, *The* $\bar{\partial}$*-equation in the multidimensional inverse scattering problem*, Russ. Math. Surv. **42** (1987), 109–180.

[PSU10] L. Päivärinta, M. Salo, G. Uhlmann, *Inverse scattering for the magnetic Schrödinger operator*, J. Funct. Anal. **259** (2010), 1771–1798.

[PS08] L. Päivärinta, J. Sylvester, *Transmission eigenvalues*, SIAM Journal on Mathematical Analysis **40** (2008), 738–753.

[Ra88] A.G. Ramm, *Recovery of the potential from fixed energy scattering data*, Inverse Problems **4** (1988), 877-886.

[Re59] T. Regge, *Introduction to complex orbital moments*, Il Nuovo Cimento **14** (1959), 951–976.

[Ru02] A. Ruiz, *Harmonic analysis and inverse problems.* Lecture notes, 2002, www.uam.es/gruposinv/inversos/publicaciones/Inverseproblems.pdf.

[Sa66] P. Sabatier, *Asymptotic properties of the potentials in the inverse-scattering problem at fixed energy*, J. Mathematical Phys. **7** (1966), 1515–1531.

[SW71] E.M. Stein, G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, 1971.

[SU87] J. Sylvester, G. Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Ann. of Math. **125** (1987), 153–169.

[Tr92] H. Triebel, Theory of function spaces II. Birkhäuser, 1992.

[Uh09] G. Uhlmann, *Electrical impedance tomography and Calderón's problem*, Inverse Problems **25** (2009), 123011.

[Uh13] G. Uhlmann (editor), *Inverse Problems and Applications: Inside Out II*, MSRI Publications, **60**, Cambridge University Press, 2013.

[UV02] G. Uhlmann, A. Vasy, *Fixed energy inverse problem for exponentially decreasing potentials*, Methods Appl. Anal. **9** (2002), 239–248.

[Ve14] E.V. Vesalainen, *Rellich type theorems for unbounded domains*, preprint (2014), arXiv:1401.4531.